Mathematics 325 | Exam II Summer 2010 | Name: Dr. Grow

1.(33 pts.) Find a solution to $u_t - u_{xx} = 0$ for $-\infty < x < \infty$, $0 < t < \infty$, which satisfies $u(x,0) = x^3$ for $-\infty < x < \infty$. You may find these identities useful:

$$\int_{-\infty}^{\infty} e^{-p^2} p\,dp = 0 = \int_{-\infty}^{\infty} e^{-p^2} p^3\,dp \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-p^2} p^2\,dp = \frac{\sqrt{\pi}}{2}.$$

A candidate for a solution is $u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \varphi(y)\,dy = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} y^3\,dy$.

5 pts to here.

However $\varphi(x) = x^3$ is not bounded on $(-\infty, \infty)$ so we must check our answer at the end to make sure it is a solution. Let $p = \frac{y-x}{\sqrt{4t}}$ in the integral. Then $dp = \frac{dy}{\sqrt{4t}}$, $y \to -\infty \Leftrightarrow p \to -\infty$, and $y \to +\infty \Leftrightarrow p \to +\infty$. Also $y = x + p\sqrt{4t}$, so

10 pts. to here

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (x + p\sqrt{4t})^3\,dp = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left(x^3 + 3x^2 p\sqrt{4t} + 3x(p\sqrt{4t})^2 + (p\sqrt{4t})^3 \right) dp$$

18 pts to here.

$$= \frac{x^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp + \frac{3x^2\sqrt{4t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p\,dp + \frac{12xt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp + \frac{4t\sqrt{4t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp.$$

22 pts. to here.

Using $\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}$ and the three given identities yields

$$u(x,t) = x^3 + \frac{12xt}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = x^3 + 6xt.$$

30 pts. to here.

Check: $u_t - u_{xx} = 6x - 6x = 0$ for all $-\infty < x < \infty$, $-\infty < t < \infty$.

$u(x,0) = x^3$ for $-\infty < x < \infty$.

33 pts to here.

$\therefore$ $\boxed{u(x,t) = x^3 + 6xt}$ is a solution.

2.(33 pts.) Use Fourier transform methods to solve $u_{xx}+u_{yy}=0$ (1) in $-\infty<x<\infty,\ 0<y<\infty$, subject to the boundary condition

$$u(x,0) \overset{(2)}{=} \begin{cases} 1 & \text{if } |x|<1, \\ 0 & \text{otherwise,} \end{cases}$$

and the decay conditions

$$\lim_{y\to\infty} u(x,y) \overset{(3)}{=} 0 \text{ for each } x \text{ in } (-\infty,\infty)$$

and

$$|u(x,y)| \overset{(4)}{\le} |g(x)| \text{ for all } x \text{ in } (-\infty,\infty) \text{ and all } y>0.$$

(Here g is an absolutely integrable function on $(-\infty,\infty)$.)

We take the Fourier transform of (1) with respect to x:

$$0 = \mathcal{F}(0)(\xi) = \mathcal{F}(u_{xx}+u_{yy})(\xi) = (i\xi)^2\mathcal{F}(u)(\xi) + \frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi)$$

4 pts. to here.

$$\Rightarrow \quad 0 = \frac{\partial^2 \mathcal{F}}{\partial y^2}(u)(\xi) - \xi^2\mathcal{F}(u)(\xi).$$

The general solution of this ODE in the independent variable y (with parameter ξ) is

8 to here

$\mathcal{F}(u)(\xi) = c_1(\xi)e^{-\xi y} + c_2(\xi)e^{\xi y}$. If $\xi>0$ then (3)-(4) imply $c_2(\xi)=0$. If $\xi<0$ then (3)-(4) imply $c_1(\xi)=0$. Consequently,

13 pts. to here

$$\mathcal{F}(u)(\xi) = \begin{cases} c_1(\xi)e^{-\xi y} & \text{if } \xi>0 \\ c_2(\xi)e^{\xi y} & \text{if } \xi<0 \end{cases} = A(\xi)e^{-|\xi|y}.$$

Applying (2) yields

17 pts. to here

$$A(\xi) = \mathcal{F}(u)(\xi)\Big|_{y=0} = \mathcal{F}(u(\cdot,0))(\xi) = \mathcal{F}(\chi_{(-1,1)})(\xi)$$

and formula C in the table of Fourier transforms (with a = y) gives

21 pts. to here.

$$e^{-|\xi|y} = \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi).$$

Therefore $\mathcal{F}(u)(\xi) = \mathcal{F}(\chi_{(-1,1)})(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi)$. Using the

(OVER)

Convolution identity: $\widehat{f*g}(\xi) = \sqrt{2\pi}\,\hat{f}(\xi)\hat{g}(\xi)$, we have

$$\mathcal{F}_1(u)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}_1\left(\chi_{(-1,1)} * \sqrt{\frac{2}{\pi}}\cdot\frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

25 pts. to here.

$$= \mathcal{F}_1\left(\frac{1}{\pi}\chi_{(-1,1)} * \frac{y}{(\cdot)^2+y^2}\right)(\xi).$$

By the inversion theorem,

$$u(x,y) = \frac{1}{\pi}\left(\chi_{(-1,1)} * \frac{y}{(\cdot)^2+y^2}\right)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\chi_{(-1,1)}(z)\frac{y}{(x-z)^2+y^2}\,dz$$

29 pts. to here.

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{y}{(x-z)^2+y^2}\,dz = \frac{1}{\pi}\cdot\frac{1}{y}\int_{-1}^{1}\frac{1}{\left(\frac{x-z}{y}\right)^2+1}\,dz.$$ Let $v = \frac{x-z}{y}$. Then

$dv = -\frac{dz}{y}$, $z=-1 \Leftrightarrow v = \frac{x+1}{y}$, and $z=1 \Leftrightarrow v = \frac{x-1}{y}$. Thus

$$u(x,y) = \frac{1}{\pi}\int_{\frac{x-1}{y}}^{\frac{x+1}{y}}\frac{1}{v^2+1}\,dv = \frac{1}{\pi}\left(\operatorname{Arctan}(v)\right)\Big|_{\frac{x-1}{y}}^{\frac{x+1}{y}}$$

33 pts. to here.

$$\boxed{u(x,y) = \frac{1}{\pi}\left(\operatorname{Arctan}\left(\frac{x+1}{y}\right) - \operatorname{Arctan}\left(\frac{x-1}{y}\right)\right)}.$$

3.(34 pts.) Find a solution to the damped wave equation

(1) $$u_{tt} - u_{xx} + 2u_t = 0 \text{ in } 0 < x < \pi,\ 0 < t < \infty,$$

satisfying

(2)-(3) $$u_x(0,t) = 0 = u_x(\pi, t) \text{ for } t \geq 0,$$

(4) $$u(x,0) = 0 \text{ for } 0 \leq x \leq \pi,$$

and

(5) $$u_t(x,0) = \cos^2(x) \text{ for } 0 \leq x \leq \pi.$$

Bonus (15): Show that the solution to (1)-(2)-(3)-(4)-(5) is unique.

We use separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem; (1)-(2)-(3)-(4), of the form $u(x,t) = X(x)T(t)$. Then (1) becomes $X(x)T''(t) - X''(x)T(t) + 2X(x)T'(t) = 0$. Dividing both sides of this equation by $u(x,t) = X(x)T(t)$ and rearranging produces

3 pts. to here

$$-\frac{X''(x)}{X(x)} = -\left(\frac{T''(t)}{T(t)} + \frac{2T'(t)}{T(t)}\right) = \text{constant} = \lambda.$$

Applying (2), (3), and (4), we obtain the coupled system:

7 pts. to here.

$$\begin{cases} \boxed{X''(x) + \lambda X(x) = 0,\quad X'(0) = 0 = X'(\pi)} \\ T''(t) + 2T'(t) + \lambda T(t) = 0,\quad T(0) = 0. \end{cases}$$

Eigenvalue Problem

13 pts. to here

From Section 4.2, the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2$ $(n = 0, 1, 2, \dots)$ and the corresponding eigenfunctions are $X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right) = \cos\left(\frac{n\pi x}{\pi}\right) = \cos(nx)$ $(n = 0, 1, 2, \dots)$.

The associated equation in t becomes

$$T_n''(t) + 2T_n'(t) + n^2 T_n(t) = 0.$$

$T_n(t) = e^{r_n t}$ leads to $r_n^2 + 2r_n + n^2 = 0$ and its solutions are $r_n = \frac{-2 \pm \sqrt{4 - 4n^2}}{2}$ $= -1 \pm \sqrt{1 - n^2}$. Hence we have the following cases and their general solutions.

$\underline{n=0}$: $r_0 = -1 \pm \sqrt{1} = -2$ or 0 so $T_0(t) = c_1 + c_2 e^{-2t}$.

16 pts to here

$\underline{n=1}$: $r_1 = -1 \pm \sqrt{0} = -1$ (multiplicity 2) so $T_1(t) = c_1 e^{-t} + c_2 t e^{-t}$.

19 pts. to here.

$\underline{n \geq 2}$: $r_n = -1 \pm i\sqrt{n^2 - 1}$ so $T_n(t) = c_1 e^{-t}\cos\left(t\sqrt{n^2-1}\right) + c_2 e^{-t}\sin\left(t\sqrt{n^2-1}\right)$

(OVER)

Applying $T_n(0)=0$ yields (up to a constant factor):

$$T_0(t) = 1-e^{-2t}, \quad T_1(t) = te^{-t}, \quad \text{and} \quad T_n(t) = e^{-t}\sin\left(t\sqrt{n^2-1}\right) \text{ for } n\geq 2.$$

Therefore

$$u_0(x,t) = X_0(x)T_0(t) = 1-e^{-2t}$$

21 pts. to here

$$u_1(x,t) = X_1(x)T_1(t) = te^{-t}\cos(x)$$

$$u_n(x,t) = X_n(x)T_n(t) = e^{-t}\sin\left(t\sqrt{n^2-1}\right)\cos(nx) \quad (n=2,3,4,\dots)$$

are solutions to (1)-(2)-(3)-(4). By the superposition principle, a formal solution to (1)-(2)-(3)-(4) is

$$u(x,t) = c_0(1-e^{-2t}) + c_1 te^{-t}\cos(x) + \sum_{n=2}^{\infty} c_n e^{-t}\sin\left(t\sqrt{n^2-1}\right)\cos(nx)$$

24 pts. to here.

for arbitrary constants $c_0, c_1, c_2, \dots$ To satisfy (5), we need

$$u_t(x,t) = 2c_0e^{-2t} + c_1(1-t)e^{-t}\cos(x) + \sum_{n=2}^{\infty} c_n\left(\sqrt{n^2-1}\cos\left(t\sqrt{n^2-1}\right)-\sin\left(t\sqrt{n^2-1}\right)\right)e^{-t}\cos(nx)$$

We want

$$\tfrac{1}{2}+\tfrac{1}{2}\cos(2x) = \cos^2(x) = u_t(x,0) = 2c_0 + c_1\cos(x) + \sum_{n=2}^{\infty} c_n\left(\sqrt{n^2-1}\right)\cos(nx)$$

27 pts. to here.

for all $0\leq x\leq\pi$. By inspection, we may take $c_0=\frac{1}{4}$, $c_2=\frac{1}{2\sqrt{3}}$, and $c_n=0$

30 pts. to here.

for all other n-values. That is,

$$\boxed{u(x,t) = \frac{1}{4}\left(1-e^{-2t}\right) + \frac{1}{2\sqrt{3}}e^{-t}\sin\left(t\sqrt{3}\right)\cos(2x)}$$

33 pts. to here.

solves (1)-(2)-(3)-(4)-(5).

<u>Bonus</u>: Let $v(x,t)$ be another solution to (1)-(2)-(3)-(4)-(5) and consider $w(x,t) = u(x,t)-v(x,t)$ where $u=u(x,t)$ is the solution in

pt. to here

(CONT.)

the preceding boxed equation. Then w satisfies

$$(6) \qquad w_{tt} - w_{xx} + 2w_t = 0 \quad \text{in } 0<x<\pi,\ 0<t<\infty,$$

$$(7)-(8) \qquad w_x(0,t) = 0 = w_x(\pi,t) \quad \text{for } t \geq 0,$$

$$(9) \qquad w(x,0) = 0 \quad \text{for } 0 \leq x \leq \pi,$$

4 pts. to here

$$(10) \qquad w_t(x,0) = 0 \quad \text{for } 0 \leq x \leq \pi.$$

Consider the energy function for w given by

6 pts. to here

$$E(t) = \int_0^\pi \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t)\right] dx \qquad (t \geq 0).$$

Then

7 pts. to here

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^\pi \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t)\right] dx = \int_0^\pi \frac{\partial}{\partial t}\left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t)\right] dx$$

ts. to here.

$$= \int_0^\pi \left[w_t(x,t) w_{tt}(x,t) + \overbrace{w_x(x,t)}^{U}\,\overbrace{w_{tx}(x,t)}^{dV}\right] dx = \int_0^\pi w_t(x,t) w_{tt}(x,t)\,dx + w_x(x,t) w_t(x,t)\Big|_{x=0}^{\pi} -$$

9 pts. to here

$\int_0^\pi w_t(x,t) w_{xx}(x,t)\,dx$. By (7)-(8), $w_x(x,t) w_t(x,t)\Big|_{x=0}^{\pi} = 0$ so

10 pts. to here.

$$\frac{dE}{dt} = \int_0^\pi w_t(x,t)\left[w_{tt}(x,t) - w_{xx}(x,t)\right] dx = \int_0^\pi w_t(x,t)\left[-2 w_t(x,t)\right] dx \quad \text{by (6).}$$

11 pts. to here.

But then $\dfrac{dE}{dt} = -2\int_0^\pi w_t^2(x,t)\,dx \leq 0$ so $E = E(t)$ is nonincreasing on $0 \leq t < \infty$

Hence

12 pts. to here.

$$0 \leq E(t) = \int_0^\pi \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t)\right] dx \leq E(0) = \int_0^\pi \left[\tfrac{1}{2} w_t^2(x,0) + \tfrac{1}{2} w_x^2(x,0)\right] dx = 0$$

13 pts. to here

by (10)-(9). By the vanishing theorem, $w_t(x,t) = 0 = w_x(x,t)$ for all $0 \leq x \leq \pi$

14 pts. here.

and all $t \geq 0$; i.e. $w(x,t) = \text{constant}$ in $0 \leq x \leq \pi$, $0 \leq t < \infty$. But (9) implies

15 pts. to here.

this constant must be zero. Therefore $u(x,t) - v(x,t) = w(x,t) = 0$ for $0 \leq x \leq \pi$

and $t \geq 0$. This is the desired conclusion: $v = u$.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2 \sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi - a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{i(\xi - a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert < a. \end{cases}$