Mathematics 325 Exam III Fall 2010 Name: Dr. Grow

1.(33 pts.) (12) (a) Show that the Fourier sine series $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$ of $f(x) = 4x(1-x)$ on $[0,1]$ is

$$4x(1-x) \sim \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\pi x)}{(2k-1)^3} .$$

(3) (b) On the same coordinate axes, sketch the graph of $f$ and the third Fourier sine partial sum

$$S_3 f(x) = \sum_{n=1}^{3} b_n \sin(n\pi x) .$$

(6) (c) Show that the Fourier sine series of $f$ converges uniformly to $f$ on the interval $[0,1]$.

(6) (d) Discuss the pointwise convergence and $L^2$ – convergence of the Fourier sine series of $f$. Give reasons for your answers.

(6) (e) Use the results of the previous parts of this problem to find the sums of the series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} .$$

(a) $$b_n = \frac{\langle f, \sin(n\pi\cdot)\rangle}{\langle \sin(n\pi\cdot), \sin(n\pi\cdot)\rangle} = \frac{\int_0^1 4x(1-x)\sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} = 2\int_0^1 \overbrace{4x(1-x)}^{u}\,\overbrace{\sin(n\pi x)\,dx}^{dv}$$

$$= \cancel{8x(1-x)\left(\frac{-\cos(n\pi x)}{n\pi}\right)\Big|_0^1}^{\,0} - \int_0^1 (8-16x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)dx = \frac{1}{n\pi}\int_0^1 \overbrace{(8-16x)}^{u}\,\overbrace{\cos(n\pi x)\,dx}^{dv}$$

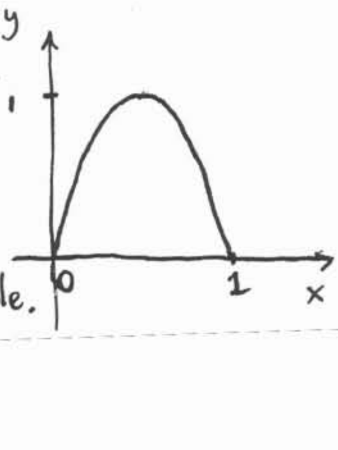
$$= \cancel{\frac{1}{n\pi}(8-16x)\left(\frac{\sin(n\pi x)}{n\pi}\right)\Big|_0^1}^{\,0} - \frac{1}{n\pi}\int_0^1 -16\,\frac{\sin(n\pi x)}{n\pi}\,dx = \frac{16}{(n\pi)^2}\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1 = \frac{16\left[1-(-1)^n\right]}{(n\pi)^3}$$

$$\therefore\ b_n = \begin{cases} 0 & \text{if } n = 2k \text{ is even,} \\ \dfrac{32}{[\pi(2k-1)]^3} & \text{if } n = 2k-1 \text{ is odd.} \end{cases}$$

Hence the Fourier sine series of $f$ is

$$4x(1-x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \boxed{\frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\pi x)}{(2k-1)^3}} .$$

(b) $$S_3 f(x) = \sum_{n=1}^{3} b_n \sin(n\pi x) = \frac{32}{\pi^3}\left(\sin(\pi x) + \frac{1}{27}\sin(3\pi x)\right).$$




The graphs of $f$ and $S_3 f$ on $0 \le x \le 1$ are nearly indistinguishable.

(c) Note that $\Phi = \{\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots\}$ is the complete orthogonal set of eigenfunctions for $X''(x) + \lambda X(x) = 0$ in $0<x<1$, subject to the Dirichlet boundary conditions $X(0)=0$, $X(1)=0$. Also observe that

(i) $f(x) = 4x(1-x)$, $f'(x) = 4-8x$, and $f''(x) = -8$ exist and are continuous for $0 \le x \le 1$, and

(ii) $f(0)=0$, $f(1)=0$ so $f$ satisfies the given (symmetric) Dirichlet boundary conditions.

By Theorem 2, the Fourier sine series of $f$ converges uniformly to $f$ on $[0,1]$.

(d) Since the uniform convergence of a sequence of functions on a bounded interval implies both the pointwise and the $L^2$-convergence of that sequence of functions on the same interval, it follows from part (c) that the Fourier sine series of $f$ at $x$ converges pointwise to $f(x)$ for all $x$ in $[0,1]$ and the Fourier sine series of $f$ converges to $f$ in the $L^2$-sense on $[0,1]$.

(e) By part (d) we have $1 = 4\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{32}{\pi^3}\sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi/2)}{(2k-1)^3} = \frac{32}{\pi^3}\sum_{k=1}^{\infty} \frac{(-1)}{(2k}$

and it follows that $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \boxed{\frac{\pi^3}{32}} \doteq 0.968946146259$.

Since $\Phi = \{\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots\}$ is a complete orthogonal set on $[0,1]$, Parseval's identity

$$\sum_{n=1}^{\infty} |b_n|^2 \int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 |f(x)|^2\,dx$$

applied to $f(x) = 4x(1-x)$ gives $\displaystyle\sum_{k=1}^{\infty}\left(\frac{32}{\pi^3(2k-1)^3}\right)^2 \int_0^1 \sin^2((2k-1)\pi x)\,dx = \int_0^1 (4x(1-x))^2\,dx$

Therefore $\displaystyle\frac{512}{\pi^6}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} = 16\int_0^1 (x^2-2x^3+x^4)\,dx = 16\left(\frac{x^3}{3}-\frac{x^4}{2}+\frac{x^5}{5}\right)\Big|_0^1 = \frac{16}{30} = \frac{8}{15}$

Consequently $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} = \boxed{\frac{\pi^6}{960}} \doteq 1.00144707664$.

2.(34 pts.) Find a solution of $u_t - t^2 u_{xx} - u \overset{①}{=} 0$ in the open strip $0 < x < 1$, $0 < t \leq \infty$, such that $u$ is continuous on the closure of the strip, satisfies the boundary conditions $u(0,t) \overset{②}{=} 0 \overset{③}{=} u(1,t)$ for $t \geq 0$, and satisfies the initial condition $u(x,0) \overset{④}{=} 4x(1-x)$ for $0 \leq x \leq 1$. (Note: You may find useful the results of problem 1.)

We use the method of separation of variables. We look for nontrivial solutions of ①-②-③
3 pts. to here of the form $u(x,t) = X(x)T(t)$. Substituting in ① yields $X(x)T'(t) - t^2 X''(x)T(t) - X(x)T(t) =$
and dividing through by $X(x)T(t)$ produces $\frac{T'(t)}{T(t)} - t^2 \frac{X''(x)}{X(x)} - 1 = 0$. Rearranging we have

6 pts. to here. ⑤ $$-\frac{X''(x)}{X(x)} = \frac{1}{t^2}\left(1 - \frac{T'(t)}{T(t)}\right) = \text{constant} = \lambda.$$

Substituting $u(x,t) = X(x)T(t)$ in ② and ③ and using the fact that we seek solutions that are
9 pts. to here not identically zero leads to $X(0) = 0 = X(1)$. Combining this with ⑤ we have the system

12 pts. to here ⑥ $$\begin{cases} X''(x) + \lambda X(x) = 0, \quad X(0) = 0 = X(1), \\ T'(t) + (\lambda t^2 - 1)T(t) = 0. \end{cases}$$

The first line in ⑥ is a standard eigenvalue problem for the operator $-\frac{d^2}{dx^2}$ with Dirichlet
16 pts. to here boundary conditions. The eigenvalues are $\lambda_n = (n\pi)^2$ and the eigenfunctions are $X_n(x) = \sin(n\pi x)$
where $n = 1, 2, 3, \ldots$ (See Sec. 4.1.) Therefore the second line in ⑥ becomes $T_n'(t) + (n^2\pi^2 t^2 - 1)T_n(t) = 0$.
Separating variables in this ODE yields

19 pts. to here $$\ln(T_n(t)) = \int \frac{T_n'(t)}{T_n(t)}\,dt = \int (1 - n^2\pi^2 t^2)\,dt = t - \frac{n^2\pi^2 t^3}{3} + c$$

22 pts. to here so $T_n(t) = A e^{t - \frac{n^2\pi^2 t^3}{3}}$ where $A = e^c$. The superposition principle gives the formal solution to ①-②-③:

25 pts. to here. $$u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{t - \frac{n^2\pi^2 t^3}{3}}.$$

We need to choose the arbitrary constants $b_1, b_2, b_3, \ldots$ so ④ is satisfied:

pts to here. $$4x(1-x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\pi n x) \quad \text{for } 0 \leq x \leq 1.$$

31 pts to here

By problem 1, we should take $b_{2k-1} = \frac{32}{\pi^3(2k-1)^3}$ and $b_{2k} = 0$ for $k = 1, 2, 3, \ldots$

Therefore

34 pts. to here.

$$\boxed{u(x,t) = \frac{32}{\pi^3} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi x)\, e^{t - \frac{(2k-1)^3 \pi^3 t^3}{3}}}{(2k-1)^3}}$$

solves ①-②-③-④.

30

3.(33 pts.) (a) Find a solution of $u_{xx} + u_{yy} + u_{zz} = x^2 + y^2 + z^2$ in the open spherical shell $1 < \sqrt{x^2 + y^2 + z^2} < 2$ such that $u$ is continuous on the closure of the shell and satisfies the boundary conditions $u(x, y, z) = \frac{21}{20}$ if $\sqrt{x^2 + y^2 + z^2} = 1$ and $u(x, y, z) = \frac{23}{10}$ if $\sqrt{x^2 + y^2 + z^2} = 2$.

3 (b) Is the solution to the problem in part (a) unique? Justify your answer.

(a) In spherical coordinates the problem becomes

5 pts. to here

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2 \sin(\varphi)}\dfrac{\partial}{\partial \varphi}\left(\sin(\varphi)\dfrac{\partial u}{\partial \varphi}\right) + \dfrac{1}{r^2 \sin^2(\varphi)}\dfrac{\partial^2 u}{\partial \theta^2} = r^2 \\ u = \dfrac{21}{20} \text{ if } r = 1, \quad u = \dfrac{23}{10} \text{ if } r = 2. \end{cases}$$

Because the PDE, the region, and the boundary conditions are invariant under rotations, we will assume that the solution $u = u(r, \theta, \varphi)$ is a function of $r$ alone, independent of the angles $\theta$ and $\varphi$. Then $\frac{\partial u}{\partial \varphi} = 0 = \frac{\partial u}{\partial \theta}$ so the PDE reduces to an ODE:

10 pts. to here. $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = r^2$. Then $\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = r^4$ so integration gives $r^2 \frac{du}{dr} = \frac{r^5}{5} + c_1$, 15 pts. to here

and hence $\frac{du}{dr} = \frac{r^3}{5} + \frac{c_1}{r^2}$. Integrating again produces $u(r) = \frac{r^4}{20} - \frac{c_1}{r} + c_2$. 20 pts. to here

Applying the boundary conditions we have $\frac{21}{20} = u(1) = \frac{1}{20} - c_1 + c_2$ and $\frac{23}{10} = u(2) = \frac{16}{20} - \frac{c_1}{2} + c_2$. That is, $\begin{cases} 1 = -c_1 + c_2 \\ \frac{3}{2} = -\frac{c_1}{2} + c_2 \end{cases}$. Subtracting equations 25 pts. to here

yields $\frac{1}{2} = \frac{c_1}{2}$ so $c_1 = 1$. Then substituting in the first equation of the system gives $c_2 = 2$.

30 pts. to here. Consequently, $\boxed{u(r) = \frac{r^4}{20} - \frac{1}{r} + 2}$. In cartesian coordinates,

$$\boxed{u(x, y, z) = \frac{(x^2+y^2+z^2)^2}{20} - \frac{1}{\sqrt{x^2+y^2+z^2}} + 2}.$$

3 pts. (b) By the uniqueness theorem for solutions to Poisson's equation in the presence of Dirichlet boundary conditions (see text, pp. 149-150), it follows that the solution in part (a) is unique.

# Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x) + \lambda X(x) = 0 \text{ in } a < x < b \text{ with any symmetric boundary conditions}$$

and let $\Phi = \{X_1, X_2, X_3, \ldots\}$ be the complete orthogonal set of eigenfunctions for (1). Let $f$ be any absolutely integrable function defined on $a \le x \le b$. Consider the Fourier series for $f$ with respect to $\Phi$:

$$f(x) \sim \sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \quad (n = 1, 2, 3, \ldots).$$

**Theorem 2.** (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f$ satisfies the given symmetric boundary conditions,

then the Fourier series of $f$ converges uniformly to $f$ on $[a,b]$.

**Theorem 3.** ($L^2$ – Convergence) If

$$\int_a^b |f(x)|^2 \, dx < \infty$$

then the Fourier series of $f$ converges to $f$ in the mean-square sense in $(a,b)$.

**Theorem 4.** (Pointwise Convergence of Classical Fourier Series)

(i) If $f$ is a continuous function on $a \le x \le b$ and $f'$ is piecewise continuous on $a \le x \le b$, then the classical Fourier series (full, sine, or cosine) at $x$ converges pointwise to $f(x)$ in the open interval $a < x < b$.

(ii) If $f$ is a piecewise continuous function on $a \le x \le b$ and $f'$ is piecewise continuous on $a \le x \le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point $x$ in $(-\infty, \infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x) = \frac{f(x^+) + f(x^-)}{2}$$

for all $x$ in the open interval $(a,b)$.

**Theorem 4$\infty$.** If $f$ is a function of period $2l$ on the real line for which $f$ and $f'$ are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+) + f(x^-)}{2}$ for every real $x$.

## Trigonometric Identities

$$\cos^2(A) = \frac{1}{2} + \frac{1}{2}\cos(2A)$$
$$\sin^2(A) = \frac{1}{2} - \frac{1}{2}\cos(2A)$$
$$\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$
$$\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$$
$$\cos(A)\cos(B) = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$
$$\cos^3(A) = \frac{3}{4}\cos(A) + \frac{1}{4}\cos(3A)$$
$$\sin^3(A) = \frac{3}{4}\sin(A) - \frac{1}{4}\sin(3A)$$

## Expressions for the Laplacian Operator

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{(polar coordinates)}$$
$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{(cylindrical coordinates)}$$
$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\varphi)}\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2\sin^2(\varphi)}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{(spherical coordinates)}$$

## Expression for the Gradient Operator

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{1}{r\sin(\varphi)}\frac{\partial f}{\partial \theta}\hat{\theta} \qquad \text{(spherical coordinates)}$$

Math 325

Exam III

Fall 2010

number taking exam: 36

median: 82

mean: 74.8

standard deviation: 17.4

Distribution of Scores:

| Range | Graduate Grade | Undergraduate Grade | Frequency |
|---|---|---|---|
| 87–100 | A | A | 13 |
| 73 – 86 | B | B | 8 |
| 60 – 72 | C | B | 6 |
| 50 – 59 | C | C | 6 |
| 0 – 49 | F | D | 3 |