Mathematics 325 Exam III Spring 2011 Name: Dr. Grow

1.(30 pts.) Consider the operator $T = -\frac{d^2}{dx^2}$ on the space $V = \left\{ \varphi \in C^2[0,\pi] : \varphi(0) = 0 \text{ and } \frac{d\varphi}{dx}(\pi) = 0 \right\}$.

(a) Show that T is a symmetric operator on V.
(b) Are all the eigenvalues of T on V real numbers? Justify your answer.
(c) Are the eigenfunctions of T on V corresponding to distinct eigenvalues orthogonal on $[0,\pi]$? Why?
(d) Are all the eigenvalues of T on V nonnegative? Justify your answer.
(e) Compute all the eigenvalues and eigenfunctions for the operator T on V.

8 pts. (a) Let φ_1 and φ_2 belong to V. Then

$$\langle T\varphi_1, \varphi_2 \rangle = \int_0^\pi T\varphi_1(x)\overline{\varphi_2(x)}\,dx = -\int_0^\pi \underbrace{\overline{\varphi_2(x)}}_{u}\, \underbrace{\varphi_1''(x)dx}_{dv} = -\left[\overline{\varphi_2(x)}\varphi_1'(x)\right]\Big|_0^\pi - \int_0^\pi \underbrace{\overline{\varphi_2'(x)}}_{u}\,\underbrace{\varphi_1'(x)dx}_{dv}$$

$$= \left[\overline{\varphi_2'(x)}\varphi_1(x) - \overline{\varphi_2(x)}\varphi_1'(x)\right]\Big|_0^\pi - \int_0^\pi \varphi_1(x)\overline{\varphi_2''(x)}\,dx.$$

But $\overline{\varphi_2'(\pi)} = 0 = \varphi_1'(\pi)$ and $\varphi_1(0) = 0 = \overline{\varphi_2(0)}$ imply $\left[\overline{\varphi_2'(x)}\varphi_1(x) - \overline{\varphi_2(x)}\varphi_1'(x)\right]\Big|_0^\pi = 0$

$\therefore$ $\langle T\varphi_1, \varphi_2 \rangle = \int_0^\pi \varphi_1(x)\overline{\left(-\varphi_2''(x)\right)}\,dx = \langle \varphi_1, T\varphi_2 \rangle$. This shows that T is a symmetric operator on V.

2 pts (b) By Theorem 2 in Sec. 5.3, all the eigenvalues of the symmetric operator T on V are real.

2 pts. (c) By Theorem 1 in Sec. 5.3, eigenfunctions corresponding to distinct eigenvalues of the symmetric operator T on V are orthogonal on $0 \le x \le \pi$.

3 pts (d) If φ is any real-valued function belonging to V then $\varphi(x)\varphi'(x)\Big|_0^\pi = \varphi(\pi)\varphi'(\pi) - \varphi(0)\varphi'(0) = 0$. Therefore the symmetric operator T on V has no negative eigenvalues by Theorem 3 in Sec. 5.3.

(e) To find the eigenvalues and eigenfunctions for the operator T on V, we need to find all (complex) numbers λ for which there corresponds a nonzero function

φ in V satisfying $T\varphi = \lambda\varphi$. That is, we need to solve the eigenvalue problem

$$\varphi''(x) + \lambda\varphi(x) \overset{①}{=} 0, \quad \varphi(0) \overset{②}{=} 0, \quad \varphi'(\pi) \overset{③}{=} 0.$$

By parts (b) and (d) of this problem we know that λ is real and $\lambda \ge 0$.

(10) <u>Case 1</u>: $\lambda > 0$, say $\lambda = \beta^2$ where $\beta > 0$.

The general solution of ① in this case is $\varphi(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$ where c_1 and c_2 are arbitrary constants. Note that $\varphi'(x) = -\beta c_1 \sin(\beta x) + \beta c_2\cos(\beta x)$. Applying ② we find $0 = \varphi(0) = c_1\cos(0) + c_2\sin(0) = c_1$. Applying ③ yields $0 = \varphi'(\pi) = -\beta c_1 \sin(\beta\pi) + \beta c_2 \cos(\beta\pi) = \beta c_2\cos(\beta\pi)$. In order that φ be nonzero it is necessary that $c_2 \neq 0$ and hence $\cos(\beta\pi) = 0$. Since cosine vanishes only at odd multiples of $\frac{\pi}{2}$, it follows that $\beta\pi = \beta_n\pi = (2n-1)\frac{\pi}{2}$ so $\beta_n = \frac{2n-1}{2}$ $(n = 1, 2, 3, \ldots)$

Therefore

$$\boxed{\begin{aligned} \lambda_n &= \beta_n^2 = \left(\frac{2n-1}{2}\right)^2 \quad (n = 1, 2, 3, \ldots) \\ \varphi_n(x) &= \sin\left((2n-1)\frac{x}{2}\right) \quad (n = 1, 2, 3, \ldots) \end{aligned}}$$

are the eigenvalues and eigenfunctions of T on V, respectively. (See case 2 below which shows that $\lambda = 0$ is not an eigenvalue of T on V.)

(2) <u>Case 2</u>: $\lambda = 0$.

The general solution of ① in this case is $\varphi(x) = c_2 x + c_1$ where c_1 and c_2 are arbitrary constants. Note that $\varphi'(x) = c_2$. Applying ② and ③ yield $0 = \varphi(0) = c_2(0) + c_1 = c_1$ and $0 = \varphi'(\pi) = c_2$. Therefore $\varphi = 0$ so $\lambda = 0$ is not an eigenvalue of T on V.

2.(30 pts.) (a) Show that the Fourier series of the function $f(x) = x(2\pi - x)$ with respect to the orthogonal set of functions $\Phi = \{\sin((2n-1)x/2)\}_{n=1}^{\infty}$ on the interval $[0,\pi]$ is

$$\frac{32}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x/2)}{(2n-1)^3}$$

(b) Write $S_2 f(x)$, the second Fourier partial sum of the Fourier series of f with respect to Φ, and sketch the graphs of f and $S_2 f$ over the interval $0 \le x \le \pi$ on the same set of coordinate axes. (To save time, raise your hand and I will come to your seat and grade your graphs from your calculator's display.)

(c) Does the Fourier series of f with respect to Φ converge to f uniformly on $[0,\pi]$? Justify your answer.

13 pts. (a) The Fourier series of f with respect to Φ on $[0,\pi]$ is $f(x) \sim \sum_{n=1}^{\infty} b_n \sin((2n-1)x/2)$

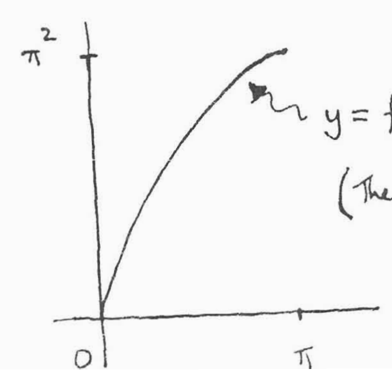
where $b_n = \dfrac{\langle f, \varphi_n\rangle}{\langle \varphi_n, \varphi_n\rangle} = \dfrac{\int_0^\pi f(x)\overline{\varphi_n(x)}\,dx}{\int_0^\pi \varphi_n(x)\overline{\varphi_n(x)}\,dx} = \dfrac{\int_0^\pi f(x)\sin((2n-1)x/2)\,dx}{\int_0^\pi \sin^2((2n-1)x/2)\,dx}$

$$= \frac{2}{\pi}\int_0^\pi \overbrace{x(2\pi - x)}^{u}\overbrace{\sin((2n-1)x/2)\,dx}^{dv} = \frac{2}{\pi}\left[\cancel{x(2\pi-x)\left(\frac{-\cos((2n-1)x/2)}{(2n-1)/2}\right)\Big|_0^\pi} + \int_0^\pi \overbrace{(2\pi-2x)}^{u}\overbrace{\frac{\cos\left(\left(\frac{2n-1}{2}\right)x\right)dx}{(2n-1)/2}}^{dv}\right]$$

$$= \frac{4}{\pi(2n-1)}\left[\cancel{\frac{2(\pi-x)\sin\left(\left(\frac{2n-1}{2}\right)x\right)}{(2n-1)/2}\Big|_0^\pi} + \int_0^\pi \frac{\sin\left(\left(\frac{2n-1}{2}\right)x\right)}{(2n-1)/2}\,2dx\right] = \frac{16}{\pi(2n-1)^2}\cdot\left(\frac{-\cos\left(\left(\frac{2n-1}{2}\right)x\right)}{(2n-1)/2}\right)\Big|_0^\pi = \boxed{\frac{32}{\pi(2n-1)^3}}$$

Thus $\boxed{f(x) \sim \dfrac{32}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin((2n-1)x/2)}{(2n-1)^3}}$ is the Fourier series of f w.r.t. Φ.

5 pts. (b) $S_2 f(x) = \dfrac{32}{\pi}\displaystyle\sum_{n=1}^{2}\frac{\sin((2n-1)x/2)}{(2n-1)^3} = \boxed{\dfrac{32}{\pi}\left(\sin\left(\frac{x}{2}\right) + \frac{1}{27}\sin\left(\frac{3x}{2}\right)\right)}$



$y = f(x) = x(2\pi - x)$

(The graph of $y = S_2 f(x)$ was indistinguishable from $y = f(x)$ on an HP-49G.)

7 pts (c) We apply Theorem 2 in Sec. 5.4. (See the attached sheet on convergence theorems for a statement of this theorem.) Note that

(2) $f(x) = x(2\pi - x)$, $f'(x) = 2\pi - 2x$, and $f''(x) = -2$ exist and are continuous for $0 \leq x \leq \pi$. Also f satisfies the symmetric boundary conditions

(3) $f(0) = 0$ and $f'(\pi) = 0$ that lead to the eigenfunctions $\Phi = \left\{ \sin\left(\left(\frac{2n-1}{2}\right)x\right) \right\}_{n=1}^{\infty}$ on $[0, \pi]$. (See problem 1.) By Theorem 2 in Sec. 5.4, the

$\boxed{\text{Fourier series of } f \text{ w.r.t. } \Phi \text{ converges uniformly to } f \text{ on } [0,\pi].}$

3.(40 pts.) Let a be a positive constant. Find a solution to the following problem. You may use the results stated in problems 1 and 2, even if you did not successfully solve those problems.

$$u_t - u_{xx} + au \overset{①}{=} 0 \quad \text{for} \quad 0 < x < \pi,\ 0 < t < \infty,$$
$$u(0,t) \overset{②}{=} 0 \quad \text{and} \quad u_x(\pi,t) \overset{③}{=} 0 \quad \text{if } 0 \le t < \infty,$$
$$u(x,0) \overset{④}{=} x(2\pi - x) \quad \text{if} \quad 0 \le x \le \pi.$$

Bonus (10 pts.): Is there at most solution to the problem above that is continuous on the strip $0 \le x \le \pi,\ 0 \le t < \infty$? Justify your answer.

ts. to here. We use the method of separation of variables. Let $u(x,t) = X(x)T(t)$ be a nontrivial solution of the homogeneous portion of this problem: ①-②-③. Substituting in ① yields $X(x)T'(t) - X''(x)T(t) + aX(x)T(t) = 0$. Rearranging yields $-\frac{X''(x)}{X(x)} = -\frac{T'(t)}{T(t)} - a =$ constant $= \lambda$. Substituting $u(x,t) = X(x)T(t)$ into ② and ③ yields

$X(0)T(t) = 0$ and $X'(\pi)T(t) = 0$ for all $t \ge 0$. Since $X(0) \ne 0$ implies $u(x,t) = X(x)T(t) = 0$ for all $0 \le x \le \pi$ and $t \ge 0$, this is impossible; u was assumed to be nontrivial.

Therefore $X(0) = 0$. A similar argument shows $X'(\pi) = 0$. Summarizing, we have the system:

8 pts. to here

$$\begin{cases} X''(x) + \lambda X(x) \overset{⑤}{=} 0, \quad X(0) \overset{⑥}{=} 0 \text{ and } X'(\pi) \overset{⑦}{=} 0, \\ T'(t) + (a+\lambda)T(t) \overset{⑧}{=} 0. \end{cases}$$

By problem 1, the eigenvalues of ⑤-⑥-⑦ are $\lambda_n = \left(\frac{2n-1}{2}\right)^2$ and the corresponding eigenfunctions are $X_n(x) = \sin\left(\left(\frac{2n-1}{2}\right)x\right)$ $(n = 1, 2, 3, \ldots)$. The solution of ⑧ when $\lambda = \lambda_n$ is $T_n(t) = e^{-[a+\lambda_n]t}$ (up to a constant multiple) so

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\left(\frac{2n-1}{2}\right)x\right)e^{-at}e^{-\left(\frac{2n-1}{2}\right)^2 t} \quad (n = 1, 2, 3, \ldots) \text{ solves ①-②-③.}$$

The superposition principle shows that

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\left(\frac{2n-1}{2}\right)x\right)e^{-at} \cdot e^{-\left(\frac{2n-1}{2}\right)^2 t}$$

is a formal solution to ①-②-③ for any choice of constants $b_1, b_2, b_3, \ldots$

To satisfy ④ we need

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$$x(2\pi - x) = f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\left(\frac{2n-1}{2}\right)x\right)$$

37 for all $0 \le x \le \pi$. By problem 2 we should choose $b_n = \dfrac{32}{\pi(2n-1)^3}$ for $n = 1, 2, 3, \ldots$

Therefore

40 pts to here.
$$\boxed{u(x,t) = \frac{32e^{-at}}{\pi}\sum_{n=1}^{\infty} \frac{e^{-(n-\frac{1}{2})^2 t}\sin\left(\left(n-\frac{1}{2}\right)x\right)}{(2n-1)^3}}$$

solves ①-②-③-④.

10 pts. **Bonus**: Yes, there is only one $\overbrace{\text{solution}}^{\text{continuous}}$ to ①-②-③-④. To see this suppose that $u = u_1(x,t)$ and $u = u_2(x,t)$ are $\overbrace{\text{solutions}}^{\text{continuous}}$ to ①-②-③-④ and consider

here. $v(x,t) = u_1(x,t) - u_2(x,t)$. Then v is a $\overbrace{\text{solution}}^{\text{continuous}}$ to the system:

$$\begin{cases} v_t - v_{xx} + av \overset{⑨}{=} 0 & \text{for } 0 < x < \pi,\ 0 < t < \infty, \\ v(0,t) \overset{⑩}{=} 0 \text{ and } v_x(\pi,t) \overset{⑪}{=} 0 & \text{for } 0 \le t < \infty \\ v(x,0) \overset{⑫}{=} 0 & \text{for } 0 \le x \le \pi. \end{cases}$$

Consider the energy function of v:

$$E(t) = \left(\int_0^{\pi} v^2(x,t)\,dx\right)^{1/2} \qquad (t \ge 0).$$

Then $$2E(t)\frac{dE}{dt} = \frac{d}{dt}\int_0^{\pi} v^2(x,t)\,dx = \int_0^{\pi} \frac{\partial}{\partial t}\left(v^2(x,t)\right)dx = \int_0^{\pi} 2v(x,t)v_t(x,t)\,dx$$

Substituting for $\underbrace{v_t}^{}$ $\overset{\text{using ⑨}}{}$ in the integrand in the right member of the above equation yields

6 , is. to here.

$$E(t)\frac{dE}{dt} = \int_0^\pi v(x,t)\left[v_{xx}(x,t) - av(x,t)\right]dx$$

$$= -a\int_0^\pi v^2(x,t)dx + \int_0^\pi \overbrace{v(x,t)}^{U}\overbrace{v_{xx}(x,t)dx}^{dV}$$

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$$= -a\int_0^\pi v^2(x,t)\,dx + v(x,t)v_x(x,t)\Big|_{x=0}^{\pi} - \int_0^\pi v_x^2(x,t)dx\,.$$

By (11) and (10), we have $v(x,t)v_x(x,t)\Big|_{x=0}^{\pi} = 0$. Therefore

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$$E(t)\frac{dE}{dt} = -\int_0^\pi\left[av^2(x,t) + v_x^2(x,t)\right]dx \le 0$$

and it follows that E is a decreasing function on the interval $0 \le t < \infty$.

Thus, for $t > 0$,

$$0 \le E(t) \le E(0) = \left(\int_0^\pi v^2(x,0)\,dx\right)^{1/2} = 0$$

by (12). By the vanishing theorem (and continuity of v) it follows that

pts to here $v(x,t) = 0$ for all $0 \le x \le \pi$ and $t \ge 0$. That is, $u_1(x,t) = u_2(x,t)$ for $0 \le x \le \pi$ and $t \ge 0$. This shows that the solution found in problem 3 is unique.

Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x) + \lambda X(x) = 0 \text{ in } a < x < b \text{ with any symmetric boundary conditions}$$

and let $\Phi = \{X_1, X_2, X_3, \dots\}$ be the complete orthogonal set of eigenfunctions for (1). Let f be any absolutely integrable function defined on $a \le x \le b$. Consider the Fourier series for f with respect to Φ:

$$\sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \qquad (n = 1, 2, 3, \dots).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2 \, dx < \infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b)

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a \le x \le b$ and f' is piecewise continuous on $a \le x \le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a < x < b$.

(ii) If f is a piecewise continuous function on $a \le x \le b$ and f' is piecewise continuous on $a \le x \le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty, \infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x) = \frac{f(x^+) + f(x^-)}{2}$$

for all x in the open interval (a,b)

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+) + f(x^-)}{2}$ for every real x.

Exam III

Math 325

Spring 2011

n : 23

mean : 66.9 median : 75

standard deviation : 27.6

Distribution of Scores	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87–100	A	A	8
73 – 86	B	B	4
60 – 72	C	B	1
50 – 59	C	C	3
0 – 49	F	D	7