1. (30 pts.) Consider the operator \( T = -\frac{d^2}{dx^2} \) on the space \( V = \{ \varphi \in C^2[0, \pi]; \varphi(0) = 0 \text{ and } \varphi(\pi) = 0 \} \).

(a) Show that \( T \) is a symmetric operator on \( V \).

(b) Are all the eigenvalues of \( T \) on \( V \) real numbers? Justify your answer.

(c) Are the eigenfunctions of \( T \) on \( V \) corresponding to distinct eigenvalues orthogonal on \( [0, \pi] \)? Why?

(d) Are all the eigenvalues of \( T \) on \( V \) nonnegative? Justify your answer.

(e) Compute all the eigenvalues and eigenfunctions for the operator \( T \) on \( V \).

\[ \begin{align*}
&\text{(a) Let } \varphi_1 \text{ and } \varphi_2 \text{ belong to } V. \text{ Then } \int_0^\pi \varphi_1(x) \frac{d^2}{dx^2} \varphi_2(x) \, dx = -\int_0^\pi \frac{d}{dx} \left( \varphi_1(x) \frac{d}{dx} \varphi_2(x) \right) \, dx = -\left[ \varphi_1(x) \varphi_2''(x) \right]_0^\pi - \int_0^\pi \varphi_1(x) \varphi_2''(x) \, dx.

&\text{But } \varphi_2(0) = 0 = \varphi_2'(0) \text{ and } \varphi_2(\pi) = 0 = \varphi_2(\pi) \text{ imply } \left[ \varphi_1(x) \varphi_2''(x) \right]_0^\pi = 0.

&\therefore \int_0^\pi \varphi_1(x) \varphi_2''(x) \, dx = \left< \varphi_1, \varphi_2'' \right> \text{, This shows that } T \text{ is a symmetric operator on } V.

&\text{(b) By Theorem 2 in Sec. 5.3, all the eigenvalues of the symmetric operator } T \text{ on } V \text{ are real.}

&\text{(c) By Theorem 1 in Sec. 5.3, eigenfunctions corresponding to distinct eigenvalues of the symmetric operator } T \text{ on } V \text{ are orthogonal on } 0 \leq x \leq \pi.

&\text{(d) If } \varphi \text{ is any real-valued function belonging to } V \text{ then } \varphi(x) \varphi'(x) \Big|_0^\pi = \varphi(0) \varphi'(0) - \varphi(\pi) \varphi'(\pi) = 0. \text{ Therefore the symmetric operator } T \text{ on } V \text{ has no negative eigenvalues by Theorem 3 in Sec. 5.3.}

&\text{(e) To find the eigenvalues and eigenfunctions for the operator } T \text{ on } V, \text{ we need to find all (complex) numbers } \lambda \text{ for which there corresponds a nonzero function} \]

$\varphi$ in $V$ satisfying $T\varphi = \lambda \varphi$. That is, we need to solve the eigenvalue problem

$$\varphi''(x) + \lambda \varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi'(\pi) = 0.$$ 

By parts (b) and (c) of this problem we know that $\lambda$ is real and $\lambda \geq 0$.

(10) **Case 1:** $\lambda > 0$, say $\lambda = \beta^2$ where $\beta > 0$.

The general solution of (1) is $\varphi(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$, where $c_1$ and $c_2$ are arbitrary constants. Note that $\varphi'(x) = -\beta c_1 \sin(\beta x) + \beta c_2 \cos(\beta x)$. Applying (2) we find $0 = \varphi(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$. Applying (3) yields $0 = \varphi'(\pi) = -\beta c_1 \sin(\beta \pi) + \beta c_2 \cos(\beta \pi) = \beta c_2 \cos(\beta \pi)$. In order that $\varphi$ be nonzero it is necessary that $c_2 \neq 0$ and hence $\cos(\beta \pi) = 0$. Since cosine vanishes only at odd multiples of $\pi/2$, it follows that $\beta \pi = (2n-1)\pi/2$ so $\beta_n = \frac{2n-1}{2}$ ($n=1, 3, \ldots$)

Therefore

$$\lambda_n = \beta_n^2 = \left(\frac{2n-1}{2}\right)^2 \quad (n=1, 3, \ldots)$$

$$\varphi_n(x) = \sin\left((2n-1)\frac{x}{2}\right) \quad (n=1, 3, \ldots)$$

are the eigenvalues and eigenfunctions of $T$ on $V$, respectively. (See case 2 below which shows that $\lambda = 0$ is not an eigenvalue of $T$ on $V$.)

(2) **Case 2:** $\lambda = 0$.

The general solution of (1) in this case is $\varphi(x) = c_2 x + c_1$, where $c_1$ and $c_2$ are arbitrary constants. Note that $\varphi'(x) = c_2$. Applying (2) and (3) yield

$$0 = \varphi(0) = c_2(0) + c_1 = c_1$$

and

$$0 = \varphi'(\pi) = c_2 \cdot \pi$$

Therefore $\varphi = 0$ so $\lambda = 0$ is not an eigenvalue of $T$ on $V$. 

2. (30 pts.) (a) Show that the Fourier series of the function $f(x) = x(2\pi - x)$ with respect to the orthogonal set of functions $\Phi = \left\{ \sin \left( \frac{(2n-1)x}{2} \right) \right\}_{n=1}^{\infty}$ on the interval $[0, \pi]$ is

$$\sum_{n=1}^{\infty} \frac{32}{\pi} \frac{\sin \left( \frac{(2n-1)x}{2} \right)}{(2n-1)}$$

(b) Write $S_2f(x)$, the second Fourier partial sum of the Fourier series of $f$ with respect to $\Phi$, and sketch the graphs of $f$ and $S_2f$ over the interval $0 \leq x \leq \pi$ on the same set of coordinate axes. (To save time, raise your hand and I will come to your seat and grade your graphs from your calculator's display.)

(c) Does the Fourier series of $f$ with respect to $\Phi$ converge to $f$ uniformly on $[0, \pi]$? Justify your answer.

**Solutions:**

(a) The Fourier series of $f$ with respect to $\Phi$ on $[0, \pi]$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \left( \frac{(2n-1)x}{2} \right)$$

where $b_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{\int_0^\pi f(x) \varphi_n(x) \, dx}{\int_0^\pi \varphi_n(x)^2 \, dx} = \frac{\int_0^\pi f(x) \sin \left( \frac{(2n-1)x}{2} \right) \, dx}{\int_0^\pi \sin^2 \left( \frac{(2n-1)x}{2} \right) \, dx}$

Thus,

$$f(x) \sim \frac{32}{\pi} \sum_{n=1}^{\infty} \frac{\sin \left( \frac{(2n-1)x}{2} \right)}{(2n-1)^3}$$

is the Fourier series of $f$ with respect to $\Phi$.

(b) $S_2f(x) = \frac{32}{\pi} \sum_{n=1}^{2} \frac{\sin \left( \frac{(2n-1)x}{2} \right)}{(2n-1)^3}$

$y = f(x) = x(2\pi - x)$

(The graph of $y = S_2f(x)$ was indistinguishable from $y = f(x)$ on an HP-49G.)
7 pts: (c) We apply Theorem 2 in Sec. 5.4. (See the attached sheet on convergence theorems for a statement of this theorem.) Note that

\[ f(x) = x(2\pi - x), \quad f'(x) = 2\pi - 2x, \quad \text{and} \quad f''(x) = -2 \quad \text{exist and are symmetric for } 0 \leq x \leq \pi. \quad \text{Also, } f \text{ satisfies the boundary conditions.} \]

(3) \[ f(0) = 0 \quad \text{and} \quad f'(\pi) = 0 \quad \text{that lead to the eigenfunctions } \Phi = \left\{ \sin\left(\frac{2n-1}{2}\pi x\right) \right\}_{n=1}^{\infty} \quad \text{on } [0,\pi]. \quad \text{(See problem 1.) By Theorem 2 in Sec. 5.4, the Fourier series of } f \text{ w.r.t. } \Phi \text{ converges uniformly to } f \text{ on } [0,\pi]. \]
3. (40 pts.) Let \( a \) be a positive constant. Find a solution to the following problem. You may use the results stated in problems 1 and 2, even if you did not successfully solve those problems.

\[
\begin{align*}
    u_t - u_{xx} + au &= 0 \quad \text{for} \quad 0 < x < \pi, \quad 0 < t < \infty, \\
    u(0,t) &= 0 \quad \text{and} \quad u_x(\pi,t) &= 0 \quad \text{if} \quad 0 \leq t < \infty, \\
    u(x,0) &= x(2\pi - x) \quad \text{if} \quad 0 \leq x \leq \pi.
\end{align*}
\]

Bonus (10 pts.): Is there at most solution to the problem above that is continuous on the strip \( 0 \leq x \leq \pi, \quad 0 \leq t < \infty \)? Justify your answer.

We use the method of separation of variables. Let \( u(x,t) = X(x)T(t) \) be a nontrivial solution of the homogeneous portion of this problem: \( \odot - \odot - \odot \). Substituting in \( \odot \) yields

\[
X(x)T'(t) - X''(x)T(t) + aX(x)T(t) = 0.
\]

Rearranging yields \( \frac{X''(x)}{X(x)} = \frac{-T'(t) - aT(t)}{T(t)} \) constant = \( \gamma \). Substituting \( u(x,t) = \gamma(x)T(t) \) into \( \odot \) and \( \odot \) yields

\[
X(x)T(t) = 0 \quad \text{and} \quad X'(x)T(t) = 0 \quad \text{for all} \quad t \geq 0. \quad \text{Since} \quad X(x) \neq 0 \quad \text{implies} \quad U(x,t) = \gamma(x)T(t) = 0 \quad \text{for all} \quad 0 \leq x \leq \pi \quad \text{and} \quad t \geq 0, \quad \text{this is impossible; \( u \) was assumed to be nontrivial.}
\]

Therefore \( \gamma(x) = 0 \). A similar argument shows \( \gamma(t) = 0 \). Summarizing, we have the system:

\[
\begin{align*}
    \begin{cases}
        \gamma(x) + \lambda \gamma(x) &= 0, \quad \gamma(0) = 0 \quad \text{and} \quad \gamma'(\pi) = 0, \\
        \gamma'(t) + (a + \lambda)\gamma(t) &= 0.
    \end{cases}
\end{align*}
\]

By problem 1, the eigenvalues of \( \odot - \odot - \odot \) are \( \lambda_n = \left( \frac{2n-1}{\pi} \right)^2 \) and the corresponding eigenfunctions are \( \gamma_n(x) = \sin\left( \frac{2n-1}{\pi} x \right) \) \( (n = 1, 2, 3, \ldots) \). The solution of \( \odot \) when \( \lambda = \lambda_n \) is \( \gamma_n(x) = e^{-\left(a + \lambda_n\right) t} \) (up to a constant multiple) so

\[
\gamma_n(x,t) = \gamma_n(x)T_n(t) = \sin\left( \frac{2n-1}{\pi} x \right) e^{-\left(a + \lambda_n\right) t} \quad (n = 1, 2, 3, \ldots) \quad \text{solves} \quad \odot - \odot - \odot.
\]

The superposition principle shows that

\[
\gamma(x,t) = \sum_{n=1}^{\infty} b_n \sin\left( \frac{2n-1}{\pi} x \right) e^{-\left(a + \lambda_n\right) t}
\]

is a formal solution to \( \odot - \odot - \odot \) for any choice of constants \( b_1, b_2, b_3, \ldots \).
To satisfy \(4\) we need

\[ x(2\pi-x) = f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n-1}{2}x\right) \]

for all \(0 \leq x \leq \pi\). By problem 2 we should choose \(b_n = \frac{32}{\pi (2n-1)^3}\) for \(n=1,2,3,\ldots\)

Therefore

\[
u(x,t) = \frac{32e^{-at}}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\frac{(n-\frac{1}{2})^2}{2t}}}{(2n-1)^3} \sin\left(\frac{n-\frac{1}{2}}{2}x\right)\]

solves \(1\) \(-\) \(2\) \(-\) \(3\) \(-\) \(4\).

**Bonus:** Yes, there is only one solution to \(1\) \(-\) \(2\) \(-\) \(3\) \(-\) \(4\). To see this suppose that \(u = u_1(x,t)\) and \(u = u_2(x,t)\) are solutions to \(1\) \(-\) \(2\) \(-\) \(3\) \(-\) \(4\) and consider \(v(x,t) = u_1(x,t) - u_2(x,t)\). Then \(v\) is a solution to the system:

\[
\begin{align*}
\frac{v}{t} - \nabla^2 v &= 0 \quad \text{for} \quad 0 < x < \pi, \quad 0 < t < \infty, \\
v(0,t) &= 0 \quad \text{and} \quad v(\pi,t) = 0 \quad \text{for} \quad 0 \leq t \leq \infty \\
v(x,0) &= 0 \quad \text{for} \quad 0 \leq x \leq \pi.
\end{align*}
\]

Consider the energy function of \(v\):

\[
E(t) = \left( \int_{0}^{\pi} v^2(x,t) \, dx \right)^{1/2} \quad (t > 0).
\]

Then

\[
2E(t) \frac{dE}{dt} = \frac{d}{dt} \int_{0}^{\pi} v^2(x,t) \, dx = \int_{0}^{\pi} \nabla v(x,t) \cdot \frac{\partial v(x,t)}{\partial t} \, dx = \int_{0}^{\pi} 2v(x,t) \frac{dv}{dt}(x,t) \, dx
\]

Substituting for \(\frac{dE}{dt}\) in the integrand in the right member of the above equation
yields
\[
E(t) \frac{dE}{dt} = \int_0^\pi v(x,t) \left[ v_x(x,t) - a v(x,t) \right] dx
\]
\[
= -a \int_0^\pi v^2(x,t) dx + \int_0^\pi \frac{dV}{V(x,t) v_x(x,t)} dx
\]
\[
= -a \int_0^\pi v^2(x,t) dx + v(x,t) v_x(x,t) \bigg|_{x=0}^{x=\pi} - \int_0^\pi v_x^2(x,t) dx.
\]

By (11) and (10), we have \( v(x,t) v_x(x,t) \bigg|_{x=0}^{x=\pi} = 0 \). Therefore,
\[
E(t) \frac{dE}{dt} = - \int_0^\pi [a v^2(x,t) + v_x^2(x,t)] dx \leq 0
\]
and it follows that \( E \) is a decreasing function on the interval \( 0 \leq t < \infty \).

Thus, for \( t > 0 \),
\[
0 \leq E(t) \leq E(0) = \left( \int_0^\pi v^2(x,0) dx \right)^{\frac{1}{2}} = 0
\]
by (12). By the vanishing theorem (and continuity of \( v \)) it follows that \( v(x,t) = 0 \) for all \( 0 \leq x \leq \pi \) and \( t \geq 0 \). That is, \( u_1(x,t) = u_2(x,t) \) for \( 0 \leq x \leq \pi \) and \( t \geq 0 \). This shows that the solution found in problem 3 is unique.
Convergence Theorems

Consider the eigenvalue problem

(1) \[ X''(x) + \lambda X(x) = 0 \quad \text{in} \quad a < x < b \] with any symmetric boundary conditions

and let \( \Phi = \{X_1, X_2, X_3, \ldots\} \) be the complete orthogonal set of eigenfunctions for (1). Let \( f \) be any absolutely integrable function defined on \( a \leq x \leq b \). Consider the Fourier series for \( f \) with respect to \( \Phi \):

\[ \sum_{n=1}^{\infty} A_n X_n(x) \]

where

\[ A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \quad (n = 1, 2, 3, \ldots). \]

Theorem 2. (Uniform Convergence) If

(i) \( f(x), f'(x), \) and \( f''(x) \) exist and are continuous for \( a \leq x \leq b \) and

(ii) \( f \) satisfies the given symmetric boundary conditions,

then the Fourier series of \( f \) converges uniformly to \( f \) on \([a, b]\).

Theorem 3. (\( L^2 \)-Convergence) If

\[ \int_a^b |f(x)|^2 \, dx < \infty \]

then the Fourier series of \( f \) converges to \( f \) in the mean-square sense in \((a, b)\).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If \( f \) is a continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) at \( x \) converges pointwise to \( f(x) \) in the open interval \( a < x < b \).

(ii) If \( f \) is a piecewise continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) converges pointwise at every point \( x \) in \((-\infty, \infty)\). The sum of the Fourier series is

\[ \sum_{n=1}^{\infty} A_n X_n(x) = \frac{f(x^+) + f(x^-)}{2} \]

for all \( x \) in the open interval \((a, b)\).

Theorem 4 \( \Rightarrow \). If \( f \) is a function of period \( 2l \) on the real line for which \( f \) and \( f' \) are piecewise continuous, then the classical full Fourier series converges to \( \frac{f(x^+) + f(x^-)}{2} \) for every real \( x \).
Exam III
Math 325
Spring 2011

n: 23
mean: 66.9 median: 75
standard deviation: 27.6

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