1. (33 pts.) (a) Show that the Fourier sine series of \( f(x) = x(\pi - x) \) on \([0, \pi]\) is 
\[
\sum_{k=0}^{\infty} \frac{8\sin((2k+1)x)}{\pi(2k+1)^3}.
\]
(b) On the same coordinate axes, sketch the graph of \( f \) and the sum of the first three nonzero terms of the Fourier sine series of \( f \) on \([0, \pi]\).
(c) Based on the graphs in part (b), does it appear that the Fourier sine series of \( f \) converges uniformly to \( f \) on \([0, \pi]\)?
(d) Assuming that the Fourier sine series of \( f \) converges pointwise to \( f \) on \([0, \pi]\), use the results above to find the sum of the infinite series
\[
\sum_{k=1}^{\infty} \frac{3}{2k^3} = \frac{3}{2} \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{\pi^3}{32}.
\]

(b) 
\[
y = f(x) = x(\pi - x)
\]
\[
y = S_f(x) = \frac{8}{\pi} \left( \frac{\sin(x)}{2} + \frac{\sin(3x)}{27} + \frac{\sin(5x)}{125} \right)
\]
(To the resolving power of an HP-49G, the graphs of \( f \) and \( S_f \) are indistinguishable on \([0, \pi]\).)

(c) \[\text{Yes} \] apparently \( S_f \to f \) uniformly on \([0, \pi]\) as \( N \to \infty \).

8. \[
\frac{\pi^2}{6} = f\left(\frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{8\sin((2k+1)\frac{\pi}{2})}{\pi(2k+1)^3}
\]
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}.
\]
2. (33 pts.) (a) Find a solution to \( u_{tt} - u_{xx} = 0 \) in the strip \( 0 < x < \pi \) and \( 0 < t < \infty \) satisfying the boundary conditions \( u(0,t) = 0 \) and \( u(\pi,t) = 0 \) for \( t \geq 0 \) and the initial conditions \( u(x,0) = x(\pi - x) \) and \( u_t(x,0) = 0 \) for \( 0 \leq x \leq \pi \). (Hint: You may find the results of problem 1 useful.)

(b) Is the solution to the problem in part (a) unique? Justify your answer.

(a) The use separation of variables. We seek nontrivial solutions to the homogeneous part of the problem, \( \Box \# - \# - \# - \# - \# \), of the form \( u(x,t) = \Xi(x)T(t) \). Substituting this functional form in (1) and rearranging yields \( -\frac{T''(t)}{T(t)} = -\frac{\Xi''(x)}{\Xi(x)} = \text{constant} = \lambda \). Also, substituting in \( \# \# \# \), and \( \# \# \# \) and using the fact that \( u \) is not identically zero gives \( \Xi(0) = 0 = \Xi(\pi) = T(0) \).

Therefore we are led to the coupled system of B.V.P.'s:

\[
\begin{cases}
\Xi''(x) + \lambda \Xi(x) = 0, & \Xi(0) = 0 = \Xi(\pi) \\
T''(t) + \lambda T(t) = 0, & T(0) = 0
\end{cases}
\]

From work done in class and on homework, the Dirichlet B.C.'s in the eigenvalue problem lead to:

- Eigenvalues: \( \lambda_n = n^2 \)
- Eigenfunctions: \( \Xi_n(x) = \sin(nx) \) \( n = 1, 2, 3, \ldots \)

The t-equations then become \( T''(t) + n^2 T(t) = 0, T'(0) = 0 \) with solution \( T_n(t) = \cos(nt) \) (up to a constant factor). Thus \( u_n(x,t) = \Xi_n(x)T_n(t) = \sin(nx) \cos(nt) \) solves \( \# \# \# \# \# \# \) for \( n = 1, 2, 3, \ldots \). By the superposition principle,

\( (\#) \) \( u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) \cos(nt) \)

solves \( \# \# \# \# \# \# \) formally for arbitrary constants \( b_1, b_2, b_3, \ldots \). We need to choose the constants so \( (\#) \) satisfies \( \# \), i.e.

\( x(\pi-x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \) \( \text{for all } 0 \leq x \leq \pi \).

By problem 1, we should choose \( b_n = \begin{cases} 0 & \text{if } n = 2k \text{ is even,} \\ \frac{2}{\pi(2k+1)^3} & \text{if } n = 2k+1 \text{ is odd.} \end{cases} \)

Therefore

\( u(x,t) = \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)^3} \sin((2k+1)t) \cos((2k+1)t) \) solves \( \# \# \# \# \# \# \).

(\over)
(b) Suppose there were another solution \( u = v(x,t) \) to ①-②-③-④-⑤.

Therefore \( w(x,t) = u(x,t) - v(x,t) \) would solve

① \[
\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{if} \quad 0 < x < \pi \quad \text{and} \quad 0 < t < \infty,
\]

②-③ \[
w(0,t) = 0 = w(\pi,t) \quad \text{if} \quad t \geq 0,
\]

④-⑤ \[
w(x,0) = 0 = w_x(x,0) \quad \text{if} \quad 0 \leq x \leq \pi.
\]

Let \( E(t) = \int_0^\pi \left[ \frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t) \right] dx \) be the energy of the solution to ①-②-③-④-⑤.

Then

\[
\frac{dE}{dt} = \frac{d}{dt} \int_0^\pi \left( \frac{1}{2} w_t^2 + \frac{1}{2} w_x^2 \right) dx = \int_0^\pi \frac{d}{dt} \left( \frac{1}{2} w_t^2 + \frac{1}{2} w_x^2 \right) dx = \int_0^\pi \left( w_t w_{tt} + w_x w_{xt} \right) dx
\]

① \[
= \int_0^\pi w_x w_{xt} dx + \int_0^\pi w_t w_{tt} dx = 2 \int_0^\pi w_t w_x w_{xt} dx = 2 \int_0^\pi w_t w_x w_{xt} dx
\]

By ②-③, \( w_t(\pi,t) = \lim_{k \to \infty} w(\pi,t+k) - w(\pi,t) = 0 \) and \( w_t(0,t) = \lim_{k \to \infty} w(0,t+k) - w(0,t) = 0 \).

Therefore

\[
\frac{dE}{dt} = 2 w_t(\pi,t) w_x(\pi,t) - 2 w_t(0,t) w_x(0,t) = 0
\]

Consequently \( E(t) = E(0) \) for all \( t \geq 0 \). But \( E(0) = \int_0^\pi \left( \frac{1}{2} w_t^2(x,0) + \frac{1}{2} w_x^2(x,0) \right) dx \) = 0 by ②-⑤. Therefore \( E(t) = 0 \) for all \( t \geq 0 \). By the vanishing theorem, for each \( t \geq 0 \), we have \( w_t^2(x,t) + w_x^2(x,t) = 0 \) for all \( 0 \leq x \leq \pi \). Consequently \( w_t(x,t) = 0 = w_x(x,t) \) for all \( (x,t) \) in the strip \( 0 \leq x \leq \pi, 0 \leq t < \infty \). It follows that \( w(x,t) = \text{constant} \) in the strip. But ②, ③, or ④ implies \( w(x,t) = 0 \) in the strip. That is, \( u(x,t) = v(x,t) \) in the strip so the solution to ①-②-③-④-⑤ is unique.
3. (33 pts.) Consider the operator \( T = \frac{d^2}{dx^2} \) on the space

\[ V = \{ f \in C^4[0,1]: f(0) = 0 = f(1) \text{ and } f''(0) = f''(1) \} \].

(a) Is \( T \) symmetric on \( V \) equipped with the inner product \( \langle f, g \rangle = \int_0^1 f(x)g(x)\,dx \)? Justify your answer.

(b) Are all the eigenvalues of \( T \) real? Justify your answer.

(c) Are eigenfunctions of \( T \), corresponding to distinct eigenvalues, orthogonal? Justify your answer.

(d) Are all the eigenvalues of \( T \) nonnegative? Justify your answer.

Bonus (10 pts.): Determine all the eigenvalues and eigenfunctions of \( T \).

---

(a) Let \( f \) and \( g \) belong to \( V \). Then

\[
\langle Tf, g \rangle = \int_0^1 \frac{d^2}{dx^2}(f(x))\,g(x)\,dx = \frac{d^2}{dx^2}\int_0^1 f(x)g(x)\,dx = \frac{d^2}{dx^2}\int_0^1 \frac{d^2}{dx^2}(f(x))\,g(x)\,dx = \int_0^1 \frac{d^2}{dx^2}(f(x))\,g(x)\,dx = \int_0^1 \frac{d^2}{dx^2}(f(x))\,g(x)\,dx = \int_0^1 \frac{d^2}{dx^2}(f(x))\,g(x)\,dx.
\]

But \( g(0) = g(1) \), \( f''(0) = f''(1) \), \( g''(0) = g''(1) \), and \( f(0) = 0 = f(1) \) so

\[
\langle Tf, g \rangle = \int_0^1 \frac{d^2}{dx^2}(f(x))\,g(x)\,dx = \langle f, Tg \rangle.
\]

Therefore \( T \) is symmetric on \( V \).

(b) [Yes] by Theorem 2 in the lecture notes for Sec. 5.3, the eigenvalues of \( T \) are real.

(c) [Yes] by Theorem 1 in the lecture notes for Sec. 5.3, eigenfunctions of \( T \) corresponding to distinct eigenvalues are orthogonal on \([0, 1]\).

(d) [Yes], all the eigenvalues of \( T \) are nonnegative. To see this we must adopt the proof of Theorem 3 in Sec. 5.3. (Theorem 3 itself does not apply directly because it is concerned with the operator \( -\frac{d^2}{dx^2} \).) Let \( \lambda \) be an eigenvalue of \( T = \frac{d^2}{dx^2} \) on \( V \).

Let \( f \) be an eigenfunction of \( T \) in \( V \) corresponding to \( \lambda \). By taking the real and imaginary parts of \( f \) if necessary, we may assume that \( f \) is real-valued. Then

\[
\lambda \langle f, f \rangle = \langle \lambda f, f \rangle = \langle Tf, f \rangle = \int_0^1 \frac{d^2}{dx^2}(f(x))\,f(x)\,dx = \int_0^1 \frac{d^2}{dx^2}(f(x))\,f(x)\,dx = f(x)f''(x)\bigg|_0^1 - \int_0^1 f''(x)f(x)\,dx = f(x)f''(x)\bigg|_0^1 + \int_0^1 f''(x)f(x)\,dx.
\]

But \( f \in V \) so \( f(0) = 0 = f(1) \) and \( f''(0) = f''(1) \) so

(over)
\[ \lambda < f, f > = \int_0^1 (f''(x))^2 \, dx \geq 0. \]

Since \( < f, f > > 0 \) as well, it follows that \( \lambda \geq 0 \). (A little more thought shows that actually \( \lambda > 0 \).)

**Bonus:** Let \( \lambda \) be a (necessarily positive real) number and \( f \) a nonzero function in \( V \) such that \( T f = \lambda f \). Since \( \lambda > 0 \) we may write \( \lambda = \beta^2 \) where \( \beta > 0 \). Then

\[
f'(x) - \beta^2 f(x) = 0, \quad \text{if } 0 < x < 1 \quad \text{and} \quad f(0) = f(1), \quad f''(0) = f''(1).
\]

Substituting \( f(x) = e^{\frac{mx}{2}} \) in (1) yields \( m^2 - \beta^2 = 0 \Rightarrow (m^2 - \beta^2)(x^2 + \beta^2) = 0 \Rightarrow m = \pm \beta, \pm i \beta \).

Therefore \( f_1(x) = \cos(\beta x), f_2(x) = \sin(\beta x), f_3(x) = \cosh(\beta x), f_4(x) = \sinh(\beta x) \) form a fundamental set of solutions of (1) since their Wronskian is \( W(f_1, f_2, f_3, f_4)(x) = \beta^6 \neq 0 \) for all real \( x \).

I.e. \( f(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x) + c_3 \cosh(\beta x) + c_4 \sinh(\beta x) \) is the general solution of (1).

Note that \( f''(x) = \beta^2 \cos(\beta x) - \beta^2 \sin(\beta x) + \beta^2 \cosh(\beta x) + \beta^2 \sinh(\beta x) \). Then

\[
0 = f(0) = c_1 + c_3 \\
0 = f''(0) = -c_1 \beta^2 + c_3 \beta^2 = (-c_1 + c_3) \beta^2 \Rightarrow c_1 = c_3 = 0.
\]

Therefore \( f(x) = c_2 \sin(\beta x) + c_4 \sinh(\beta x) \) so

\[
0 = f(1) = c_2 \sin(\beta) + c_4 \sinh(\beta) \Rightarrow 0 = 2c_2 \beta \sin(\beta)
\]

and

\[
0 = f''(1) = -c_2 \beta^2 \sin(\beta) + c_4 \beta^2 \sinh(\beta) \Rightarrow 0 = 2c_4 \beta^2 \sinh(\beta).
\]

Since \( \beta > 0 \) and \( \sin(\beta) > 0 \) it follows that \( c_4 = 0 \). In order for \( f \) to be nonzero we must therefore have \( \sin(\beta) = 0 \), so \( \beta = n \pi \) \((n = 1, 2, 3, \ldots)\).

**Eigenvectors of \( T \):** \( \lambda_n = \beta_n^2 = \frac{n \pi^2}{1} \)

**Corresponding Eigenfunctions:** \( f_n(x) = \sin(\frac{n \pi x}{L}) = \sin(n \pi x) \)

\( n = 1, 2, 3, \ldots \)
Math 325
Exam III
Summer 2009

number taking exam: 16
standard deviation: 19.5
mean: 77.5

Distribution of Scores:

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