Mathematics 325 Exam III Summer 2010 Name: Dr. Grow (1 pt.)

1.(33 pts.) Consider the second derivative operator $T = -\frac{d^2}{dx^2}$ on the vector space of functions $V = \{f \in C^2[0,1] : f(0) = 0 = f'(1)\}$ equipped with the usual inner product: $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$.

(a) Show that T is symmetric on V.

(b) Show that if f is a real-valued function in V then $f(1)f'(1) - f(0)f'(0) \le 0$.

(c) In light of (a), what can you say about the eigenvalues of T on V?

(d) In light of (b), what can you say about the eigenvalues of T on V?

(e) In light of (a), what can you say about eigenfunctions of T on V corresponding to distinct eigenvalues?

(f) Show that the eigenfunctions of T on V are, up to a constant factor, $f_n(x) = \sin((2n-1)\pi x/2)$ where $n = 1, 2, 3, \ldots$. What are the corresponding eigenvalues of T on V?

(a) Let $f \in V$ and $g \in V$. Then integrating by parts twice gives

$$\langle Tf, g\rangle = \int_0^1 -f''(x)\overline{g(x)}dx = -\overline{g(x)}f'(x)\Big|_0^1 + \int_0^1 +f'(x)\overline{g'(x)}dx$$

$$= \left(f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\right)\Big|_0^1 - \int_0^1 f(x)\overline{g''(x)}dx.$$ But $g'(1) = f'(1) = 0$ and

$f(0) = 0 = g(0)$ so $\left(f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\right)\Big|_0^1 = 0$. Thus

$\langle Tf, g\rangle = \int_0^1 f(x)\left[\overline{-g''(x)}\right]dx = \langle f, Tg\rangle$ so T is symmetric on V.

(b) If $f \in V$ then $f'(1) = 0$ and $f(0) = 0$ so $f(1)f'(1) - f(0)f'(0) = 0$.

(c) By Theorem 2 in Section 5.3, $\boxed{\text{the eigenvalues of } T \text{ are real numbers.}}$

(d) By Theorem 3 in Section 5.3, $\boxed{\text{the eigenvalues of } T \text{ are nonnegative.}}$

(e) By Theorem 1 in Section 5.3, the eigenfunctions of T corresponding to distinct eigenvalues are $\boxed{\text{orthogonal on } [0,1].}$

(f) Suppose λ is an eigenvalue of T on V and let f be a nonzero function in V such that $Tf = \lambda f$. Then f must satisfy $f''(x) + \lambda f(x) = 0$ in $0 < x < 1$ ①

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and $f(0) \overset{②}{=} 0$, $f'(1) \overset{③}{=} 0$. By parts (c) and (d) of this problem λ must be real and nonnegative. Hence there are two cases to consider.

Case $\lambda = 0$. Then the solution to ① is $f(x) = c_1 x + c_2$ where c_1 and c_2 are arbitrary constants. ② implies $c_2 = f(0) = 0$ and ③ implies $c_1 = f'(1) = 0$. Hence $f = 0$, the trivial solution. That is, $\lambda = 0$ is not an eigenvalue of T on V.

Case $\lambda > 0$, say $\lambda = \beta^2$ where $\beta > 0$. Then the solution to ①: $f''(x) + \beta^2 f(x) = 0$, is $f(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$ where c_1 and c_2 are arbitrary constants. ② implies $c_1 = f(0) = 0$ and ③ implies $\beta c_2 \cos(\beta) = f'(1) = 0$. We have nontrivial solutions if and only if $\cos(\beta) = 0$, that is, $\beta_n =$ odd multiple of $\frac{\pi}{2} = (2n-1)\frac{\pi}{2}$ $n = 1, 2, 3, \ldots$ Therefore the eigenvalues of T on V are

$$\boxed{\lambda_n = \beta_n^2 = (2n-1)^2 \frac{\pi^2}{4} \qquad (n = 1, 2, 3, \ldots)}$$

and the corresponding eigenfunctions of T on V are, up to a constant factor,

$$\boxed{f_n(x) = \sin\left((2n-1)\pi x/2\right) \qquad (n = 1, 2, 3, \ldots)}$$

2.(33 pts.) (a) Show that the generalized Fourier series of $f(x)=x(x-2)$ on $[0,1]$ with respect to the complete orthogonal set $\Phi=\{\sin((2n-1)\pi x/2)\}_{n=1}^{\infty}$ on $[0,1]$ is $f(x)\sim\frac{-32}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}$.

(b) Show that this generalized Fourier series converges uniformly to f on $[0,1]$.

(c) Use the results of (a) and (b) to help find the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}$.

Bonus (5 pts.): Use the results of (a) to help find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$.

(a) The Fourier series of f w.r.t. Φ on $[0,1]$ is

$$f(x) \sim \sum_{n=1}^{\infty} A_n \varphi_n(x)$$

where $A_n = \frac{\langle f,\varphi_n\rangle}{\langle \varphi_n,\varphi_n\rangle}$ $(n=1,2,3,\ldots)$. Note that

$$\langle \varphi_n,\varphi_n\rangle = \int_0^1 \sin^2((2n-1)\pi x/2)\,dx = \int_0^1\left[\frac{1}{2}-\frac{1}{2}\cos((2n-1)\pi x)\right]dx = \frac{x}{2} - \frac{\sin((2n-1)\pi x)}{2(2n-1)\pi}\Bigg|_0^1$$

$= \frac{1}{2}$. On the other hand, integrating by parts twice yields

$$\langle f,\varphi_n\rangle = \int_0^1 x(x-2)\sin((2n-1)\pi x/2)\,dx = \cancel{\frac{-2x(x-2)\cos((2n-1)\pi x/2)}{(2n-1)\pi}\Bigg|_0^1} + \int_0^1 2(x-1)\left(\frac{+2}{(2n-1)\pi}\right)\cos((2n-1)\pi x/2)\,dx$$

$$= \cancel{\frac{4(x-1)}{(2n-1)\pi}\left(\frac{2}{(2n-1)\pi}\right)\sin((2n-1)\pi x/2)\Bigg|_0^1} - \int_0^1 \frac{4}{(2n-1)\pi}\cdot\left(\frac{2}{(2n-1)\pi}\right)\sin((2n-1)\pi x/2)\,dx$$

$$= \frac{-8}{(2n-1)^2\pi^2}\left(\frac{-2}{(2n-1)\pi}\right)\cos((2n-1)\pi x/2)\Bigg|_{x=0}^{1} = \frac{-16}{(2n-1)^3\pi^3}.$$ Therefore

$$A_n = \frac{\frac{-16}{(2n-1)^3\pi^3}}{\frac{1}{2}} = \frac{-32}{(2n-1)^3\pi^3} \quad (n=1,2,3,\ldots)$$ so the Fourier series

of f w.r.t. Φ on $[0,1]$ is

$$\boxed{f(x) \sim \frac{-32}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}}.$$

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(b) We use Theorem 2 on the Convergence Theorems sheet. Note that $f(x) = x(x-2)$, $f'(x) = 2(x-1)$, and $f''(x) = 2$ exist and are continuous functions for $0 \le x \le 1$. Furthermore the symmetric boundary conditions $\Phi(0) = 0$ and $\Phi'(1) = 0$ for the eigenvalue problem $\Phi''(x) + \lambda\Phi(x) = 0$ on $0 < x < 1$ are satisfied by f:

$$f(0) = 0(0-2) = 0, \quad f'(1) = 2(1-1) = 0.$$

Therefore Theorem 2 shows that the Fourier series of f w.r.t. Φ, $\frac{-32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}$, converges uniformly to f on $[0,1]$.

(c) Since uniform convergence implies pointwise convergence we have

$$(*) \qquad x(x-2) = \frac{-32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}$$

for all x in $[0,1]$. Taking $x=1$ in $(*)$ yields

$$-1 = \frac{-32}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi/2)}{(2n-1)^3}$$

or equivalently, $\boxed{\frac{\pi^3}{32} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}}$.

Bonus: We apply parseval's identity, $\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |\Phi_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$ to the complete orthogonal system $\Phi = \{\Phi_n\}_{n=1}^{\infty} = \{\sin((2n-1)\pi x/2)\}_{n=1}^{\infty}$ and the function $f(x) = x(x-2)$ on $[0,1]$: $\sum_{n=1}^{\infty} \left|\frac{-32}{\pi^3(2n-1)^3}\right|^2 \int_0^1 \sin^2((2n-1)\pi x/2)\,dx = \int_0^1 [x(x-2)]^2 dx$

Then evaluating integrals and simplifying yields $\sum_{n=1}^{\infty} \frac{2^{10}}{\pi^6(2n-1)^6} \cdot \frac{1}{2} = \left(\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right)\Big|_0^1 =$

or $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{2^9} \cdot \frac{2^3}{15} = \frac{\pi^6}{2^6 \cdot 15} = \boxed{\frac{\pi^6}{960}}$$

3.(33 pts.) Solve $u_{tt} - u_{xx} \overset{①}{=} 0$ for $0 < x < 1$, $0 < t < \infty$, subject to the boundary conditions $u(0,t) \overset{②}{=} 0 \overset{③}{=} u_x(1,t)$ for $t \geq 0$ and the initial conditions $u(x,0) \overset{④}{=} x(x-2)$ and $u_t(x,0) \overset{⑤}{=} 0$ for $0 \leq x \leq 1$.
You may use the **results** of problems 1 and 2 even if you were not able to successfully solve them.
Bonus (10 pts.): Is the solution to the above initial-boundary value problem unique? Justify your answer.

We seek nontrivial solutions to the homogeneous portion ①-②-③-⑤ of the problem of the form $u(x,t) = X(x)T(t)$. (I.e. we use the method of separation of variables.) Substituting this functional form in ① we get $X(x)T''(t) - X''(x)T(t) = 0$ and hence

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \text{constant} = \lambda.$$

Also, substituting $u(x,t) = X(x)T(t)$ in ② and ③ yields $X(0)T(t) = 0 = X'(1)T(t)$ for all $t \geq 0$. Since $T(t) = 0$ for all $t \geq 0$ is impossible for nontrivial solutions, we must have $X(0) = 0 = X'(1)$. Arguing similarly for ⑤ we have $T'(0) = 0$. Summarizing we have

$$\boxed{X''(x) + \lambda X(x) = 0, \quad X(0) = 0 = X'(1),} \leftarrow \text{Eigenvalue Problem}$$

$$T''(t) + \lambda T(t) = 0, \quad T'(0) = 0.$$

By problem 1, the eigenvalues are $\lambda_n = \frac{(2n-1)\pi^2}{4}$ $(n = 1, 2, 3, \ldots)$ and the corresponding eigenfunctions are $X_n(x) = \sin((2n-1)\pi x/2)$ $(n = 1, 2, 3, \ldots)$. Substituting $\lambda = \lambda_n$ in the T-equation and solving we get $T_n(t) = c_1 \cos((2n-1)\pi t/2) + c_2 \sin((2n-1)\pi t/2)$. But $0 = T_n'(0)$ implies $c_2 = 0$ so, up to a constant factor, $T_n(t) = \cos((2n-1)\pi t/2)$.

Consequently,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos((2n-1)\pi t/2) \sin((2n-1)\pi x/2)$$

is a formal solution to ①-②-③-⑤ for arbitrary constants $A_1, A_2, A_3, \ldots$ To satisfy ④ we must have

$$x(x-2) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin((2n-1)\pi x/2) \quad \text{for } 0 \leq x \leq 1.$$

By problem 2, we should choose $A_n = \frac{-32}{\pi^3 (2n-1)^3}$ for $n = 1, 2, 3, \ldots$

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Therefore a solution is $\boxed{u(x,t) = \frac{-32}{\pi^3} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi t/2) \sin((2n-1)\pi x/2)}{(2n-1)^3}}$.

Bonus: The solution to problem 3 is unique. To see this we use energy considerations. Suppose there were another solution to problem 3, say $u = v(x,t)$. Then $w(x,t) = u(x,t) - v(x,t)$ solves the problem

$$\begin{aligned} w_{tt} - w_{xx} &\overset{1'}{=} 0 \quad \text{in } 0<x<1,\ 0<t<\infty, \\ w(0,t) &\overset{2'}{=} 0 \overset{3'}{=} w_x(1,t) \quad \text{for } 0 \le t < \infty, \\ w(x,0) &\overset{4'}{=} 0 \overset{5'}{=} w_t(x,0) \quad \text{for } 0 \le x \le 1. \end{aligned}$$

Consider the energy function

$$E(t) = \int_0^1 \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t)\right] dx \qquad (t \ge 0),$$

of the solution w. Then

$$(*) \qquad \frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}\left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t)\right] dx$$

$$= \int_0^1 \left[w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t)\right] dx$$

But an integration by parts and use of 3 and 5′ shows

$$(**) \quad \int_0^1 w_x(x,t) w_{xt}(x,t)\, dx = w_x(x,t) w_t(x,t)\Big|_0^1 - \int_0^1 w_t(x,t) w_{xx}(x,t)\, dx$$

$$= -\int_0^1 w_t(x,t) w_{xx}(x,t)\, dx .$$

Substituting from (**) into (*) and using 1′ gives

$$\frac{dE}{dt} = \int_0^1 \left[w_t(x,t) w_{tt}(x,t) - w_t(x,t) w_{xx}(x,t)\right] dx$$

$$= \int_0^1 w_t(x,t)\left[w_{tt}(x,t) - w_{xx}(x,t)\right] dx = 0$$

Therefore the energy function of w is constant. But

$$E(t) = E(0) = \int_0^1 \left[\frac{1}{2} w_t^2(x,0) + \frac{1}{2} w_x^2(x,0) \right] dx = 0$$

by (5') and (differentiation of) (4'). The vanishing theorem then implies

$$w_t(x,t) = 0 = w_x(x,t)$$

for all $0 \le x \le 1$ and $0 \le t < \infty$. Consequently $w(x,t) =$ constant in the upper strip: $0 \le x \le 1$, $0 \le t < \infty$. But then (4') implies $w(x,t) = 0$ in this strip; i.e. $u(x,t) = v(x,t)$ for $0 \le x \le 1$, $0 \le t < \infty$. This concludes the proof of the uniqueness of the solution to problem 3.

Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b \text{ with any symmetric boundary conditions}$$

and let $\Phi=\{X_1,X_2,X_3,\ldots\}$ be the complete orthogonal set of eigenfunctions for (1). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$f(x)\sim\sum_{n=1}^{\infty}A_nX_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle}\qquad (n=1,2,3,\ldots).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b|f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.

(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty}A_nX_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.