1. (28 pts.) (a) Verify that \( u_p(x, y) = e^y \) is a particular solution to the linear, nonhomogeneous partial differential equation \( yu_x - xu_y = \left( y^2 - x^2 \right) e^y \).
(b) Find the general solution of the linear, homogeneous partial differential equation \( yu_x - xu_y = 0 \).
(c) Find the solution of the partial differential equation \( yu_x - xu_y = \left( y^2 - x^2 \right) e^y \) satisfying the auxiliary condition \( u(x, 0) = x^6 + 1 \) for all real \( x \).

2. (28 pts.) Classify the type - elliptic, parabolic, or hyperbolic - of the partial differential equation 
\[
u x + u_y - 2u_{xy} = \sin(x - y) \]
and find, if possible, the general solution in the \( xy \)-plane.

3. (29 pts.) A homogeneous solid material occupying \( D = \{(x, y, z) \in \mathbb{R}^3 : 4 \leq x^2 + y^2 + z^2 \leq 100\} \) is completely insulated and its initial temperature at position \( (x, y, z) \) in \( D \) is \( 200 / (x^2 + y^2 + z^2) \).
(a) Write, without proof or derivation, the partial differential equation and initial/boundary conditions that completely govern the temperature \( u(x, y, z, t) \) at position \( (x, y, z) \) in \( D \) and time \( t \geq 0 \).
(b) Use the divergence theorem to help show that the heat energy \( H(t) = \iiint_D c \rho u(x, y, z, t) dV \) of the material in \( D \) at time \( t \) is a constant function of time. Here \( c \) and \( \rho \) denote the (positive, constant) specific heat and mass density, respectively, of the material in \( D \).
(c) Compute the (constant) steady-state temperature that the material in \( D \) reaches after a long time.

4. (28 pts.) Solve \( u_t - u_{xx} = 0 \) in the upper half-plane \( -\infty < x < \infty \), \( 0 < t < \infty \), subject to the initial condition \( u(x, 0) = x^2 \) if \( -\infty < x < \infty \). Note: You may find useful the identity \( \int_{-\infty}^\infty p^2 e^{-p^2} dp = \sqrt{\pi} / 2 \).

5. (29 pts.) Let \( f(x, t) = \begin{cases} \sqrt{\pi} / 2 & \text{if } |x| < |t|, \\ 0 & \text{otherwise}. \end{cases} \) Use Fourier transform methods to solve 
\( u_t - u_{xx} = f(x, t) \) in the \( xt \)-plane subject to the initial conditions \( u(x, 0) = 0 = u_t(x, 0) \) for all real \( x \).

6. (29 pts.) (a) Show that the \( 2\pi \)-periodic function determined by the formula \( f(x) = x^2 \) for \( x \) in the interval \( [-\pi, \pi] \) has full Fourier series \( \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(nx)}{n^2} \).
(b) Discuss the pointwise convergence or lack thereof for the full Fourier series of \( f \) at each point \( x \) in the interval \( [-\pi, \pi] \).
(c) Use the results of parts (a) and (b) to help find the sums of the series \( \sum_{n=1}^{\infty} (-1)^n / n^2 \) and \( \sum_{n=1}^{\infty} 1 / n^2 \).

7. (29 pts.) (a) Find a solution to the damped wave equation 
\( u_t - u_{xx} + 2u_t = 0 \) in \( 0 < x < \pi, \ 0 < t < \infty \).
satisfying the boundary conditions

(2)-(3) \[ u_x(0,t) = 0 = u_x(\pi,t) \text{ for } 0 \leq t < \infty, \]
(4) \[ u(x,0) = 0 \text{ for } 0 \leq x \leq \pi, \]
(5) \[ u_x(x,0) = x^2 \text{ for } 0 \leq x \leq \pi. \]

(b) Show that if \( u = u(x,t) \) satisfies (1)-(2)-(3) then its energy function

\[ E(t) = \frac{1}{2} \left[ \int_0^\pi \left[ u_t^2(x,t) + u_x^2(x,t) \right] dx \right] \]

is decreasing on \( 0 \leq t < \infty. \)

(c) Is the solution to the problem in part (a) unique? Justify your answer.
A Brief Table of Fourier Transforms

\[ f(x) \]

\[ \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx \]

A. \[
\begin{cases}
1 & \text{if } -b < x < b, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ \frac{\sqrt{2 \sin (b \xi)}}{\xi} \]

B. \[
\begin{cases}
1 & \text{if } c < x < d, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ \frac{e^{-i \xi c} - e^{-i \xi d}}{i \xi \sqrt{2\pi}} \]

c \quad \text{and} \quad d

C. \[
\frac{1}{x^2 + a^2} \quad (a > 0)
\]

\[ \frac{\sqrt{\pi} e^{-a|\xi|}}{\sqrt{2} \quad a} \]

D. \[
\begin{cases}
x & \text{if } 0 < x \leq b, \\
2b - x & \text{if } b < x < 2b, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ -1 + 2e^{-ib\xi} - e^{-2ib\xi} \]

E. \[
\begin{cases}
e^{-a x} & \text{if } x > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ \frac{1}{(a + i\xi) \sqrt{2\pi}} \]

F. \[
\begin{cases}
e^{a x} & \text{if } b < x < c, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ \frac{e^{(a - i\xi)c} - e^{(a - i\xi)b}}{(a - i\xi) \sqrt{2\pi}} \]

G. \[
\begin{cases}
e^{ix} & \text{if } -b < x < b, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ \frac{\sqrt{2 \sin (b (\xi - a))}}{\xi - a} \]

H. \[
\begin{cases}
e^{ix} & \text{if } c < x < d, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ \frac{e^{i\xi(c-a)} - e^{i\xi(d-a)}}{i(\xi - a) \sqrt{2\pi}} \]

I. \[ e^{-ax^2} \quad (a > 0) \]

\[ \frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)} \]

J. \[ \frac{\sin(ax)}{x} \quad (a > 0) \]

\[ \begin{cases}
0 & \text{if } |\xi| \geq a, \\
\frac{\pi}{2} & \text{if } |\xi| < a.
\end{cases} \]
Fourier Series Convergence Theorems

Consider the eigenvalue problem

\[ X''(x) + \lambda X(x) = 0 \quad \text{in} \quad a < x < b \]

with any symmetric boundary conditions of the form

\[ \begin{align*}
\alpha_1 f(a) + \beta_1 f(b) + \gamma_1 f'(a) + \delta_1 f'(b) &= 0 \\
\alpha_2 f(a) + \beta_2 f(b) + \gamma_2 f'(a) + \delta_2 f'(b) &= 0
\end{align*} \]

and let \( \Phi = \{X_1, X_2, X_3, \ldots \} \) be the complete orthogonal set of eigenfunctions for (1)-(2). Let \( f \) be any absolutely integrable function defined on \( a \leq x \leq b \). Consider the Fourier series for \( f \) with respect to \( \Phi \):

\[ \sum_{n=1}^{\infty} A_n X_n(x) \]

where

\[ A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \quad (n = 1, 2, 3, \ldots) \]

**Theorem 2.** (Uniform Convergence) If

(i) \( f(x), f'(x) \), and \( f''(x) \) exist and are continuous for \( a \leq x \leq b \) and

(ii) \( f \) satisfies the given symmetric boundary conditions,

then the Fourier series of \( f \) converges uniformly to \( f \) on \([a, b]\).

**Theorem 3.** (\( L^2 \)-Convergence) If

\[ \int_a^b |f(x)|^2 \, dx < \infty \]

then the Fourier series of \( f \) converges to \( f \) in the mean-square sense in \((a, b)\).

**Theorem 4.** (Pointwise Convergence of Classical Fourier Series)

(i) If \( f \) is a continuous function on \( a \leq x \leq b \) and \( f'' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) at \( x \) converges pointwise to \( f(x) \) in the open interval \( a < x < b \).

(ii) If \( f \) is a piecewise continuous function on \( a \leq x \leq b \) and \( f'' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) converges pointwise at every point \( x \) in \((-\infty, \infty)\). The sum of the Fourier series is

\[ \sum_{n=1}^{\infty} A_n X_n(x) = \frac{f(x^+) + f(x^-)}{2} \]

for all \( x \) in the open interval \((a, b)\).

**Theorem 4 \( \infty \).** If \( f \) is a function of period \( 2l \) on the real line for which \( f \) and \( f' \) are piecewise continuous, then the classical full Fourier series converges to \( \frac{f(x^+) + f(x^-)}{2} \) for every real \( x \).
#1. (a) \( u_p = e^{xy} \) so \((u_p)_x = ye^{xy} \) and \((u_p)_y = xe^{xy} \). Therefore

\[
y(u_p)_x - x(u_p)_y = ye^{xy} - x^2 e^{xy} = (y - x^2) e^{xy}.
\]

(b) Along characteristic curves \( \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \), solutions to \( a(x,y)u_x + b(x,y)u_y = 0 \) are constant. Therefore, the characteristic curves of \( yu_x - xu_y = 0 \) are given by \( \frac{dy}{dx} = -\frac{x}{y} \). Separating variables and integrating yields

\[
\frac{y}{2} + c_1 = \int y \, dy = -\int x \, dx = -\frac{x^2}{2} + c_2.
\]

Rearranging, \( \frac{y^2}{2} + x^2 = c \) where \( c \) is an arbitrary constant. When \( c > 0 \), the characteristic curves are circles centered at the origin. Along such a circle, solutions \( u = u(x,y) \) are constant:

\[
u(x, y(x)) = u(x, \pm \sqrt{c - x^2}) = u(0, \pm \sqrt{c}) = f(c).
\]

The general solution of \( yu_x - xu_y = 0 \) is

\[
u(x, y) = f \left( x^2 + y^2 \right)
\]

where \( f \) is a general \( C^1 \)-function of a single real variable.

(c) The general solution of \( yu_x - xu_y = (y^2 - x^2)e^{xy} \) is \( u = u_c + u_p \) where \( u_c \) is the general solution of the associated homogeneous equation \( yu_x - xu_y = 0 \) and \( u_p \) is any particular solution of the nonhomogeneous equation \( yu_x - xu_y = (y^2 - x^2)e^{xy} \). Thus \( u(xy) = f(x^2 + y^2) + e^{xy} \) is the general solution of \( yu_x - xu_y = (y^2 - x^2)e^{xy} \).

To satisfy the auxiliary condition, \( f \) must be chosen such that

\[
x + 1 = u(x, 0) = f(x^2 + 0) + e^{x\cdot0} = f(x^2) + 1
\]

for all real \( x \). Therefore \( f(w) = w^3 \) for all \( w \geq 0 \) so

\[
u(x, y) = (x^2 + y^2)^3 + e^{xy}.
\]
\#2. \[ u_{xx} - 2u_{xy} + u_{yy} = \sin(x-y) \]

\[ B^2 - 4AC = (-2)^2 - 4(1)(1) = 0. \] Therefore the equation is of \textit{parabolic type}.

The pde can be expressed in the form

\[ \left( \frac{\partial^2}{\partial x^2} - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} \right) u = \sin(x-y) \]

or

\[ (\frac{\partial}{\partial x} - \frac{\partial}{\partial y}) (\frac{\partial}{\partial x} - \frac{\partial}{\partial y}) u = \sin(x-y). \]

In order to solve the pde we make the change-of-coordinates

\[
\begin{align*}
\tilde{z} &= -(\beta x - \alpha y) = -(x-y) = x+y, \\
\eta &= \alpha x + \beta y = x-y.
\end{align*}
\]

By the chain rule, for any \( C^1 \) function \( v \) we have

\[ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial \tilde{z}} \frac{\partial \tilde{z}}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \tilde{z}} + \frac{\partial v}{\partial \eta}, \]

\[ \frac{\partial v}{\partial y} = \frac{\partial v}{\partial \tilde{z}} \frac{\partial \tilde{z}}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{\partial v}{\partial \tilde{z}} - \frac{\partial v}{\partial \eta}. \]

Consequently, as operators

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{z}} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{z}} - \frac{\partial}{\partial \eta}, \]

so \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2 \frac{\partial}{\partial \tilde{z}} \). Substituting in \((\ast)\) yields

\[ 4 \frac{\partial^2 u}{\partial \eta^2} = (2 \frac{\partial}{\partial \eta}) (2 \frac{\partial}{\partial \eta}) u = \sin(\eta). \]

Integrating twice produces

\[ \frac{\partial u}{\partial \eta} = \int \frac{1}{4} \sin(\eta) d\eta = \frac{1}{4} \cos(\eta) + C(\tilde{z}) \]

(Integrate w.r.t. \( \eta \) holding \( \tilde{z} \) fixed.)
\#2 (cont.) and 
\[ u = \int \left[ -\frac{1}{T} \cos(y) + c(y) \right] \, dy = -\frac{1}{T} \sin(y) + \eta c(y) + c(y). \]

Using the change-of-coordinates equation yields
\[ u(x,y) = -\frac{1}{T} \sin(x-y) + (x-y)f(x+y) + g(x+y) \]
where \( f \) and \( g \) are \( C^2 \)-functions of a single real variable.

\#3. (a) The pde and auxiliary conditions governing the temperature \( u(x,y,z,t) \) at position \((x,y,z)\) in \( D \) at time \( t \geq 0 \) are:

\[ \begin{align*}
\frac{\partial u}{\partial t} - k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) &= 0 & \text{if } & & + x^2+y^2+z^2 < 100 \text{ and } t > 0, \\
\nabla u \cdot \mathbf{n} &= 0 & \text{if } & & x^2+y^2+z^2 = 4 \text{ or } x^2+y^2+z^2 = 100 \text{ and } t > 0, \\
\n\left. u \right|_{t=0} &= \frac{200}{\sqrt{x^2+y^2+z^2}} & \text{if } & & t \leq x^2+y^2+z^2 \leq 100.
\end{align*} \]

(b) Let \( H(t) = \iiint_D c \rho u(x,y,z,t) \, dV \) be the heat energy of the material in \( D \) at time \( t \geq 0 \) where \( c \) and \( \rho \) are the (positive, constant) specific heat and mass density of the material in \( D \). Then
\[ \begin{align*}
\frac{dH}{dt} &= \frac{d}{dt} \iiint_D c \rho u(x,y,z,t) \, dV = \iiint_D c \rho \frac{\partial u}{\partial t} (x,y,z,t) \, dV = \\
\iiint_D c \rho k \nabla^2 u \, dV &= \iiint_D c \rho k \nabla u \cdot \mathbf{n} \, dS = \iiint_{\partial D} \mathbf{n} \cdot dS' = 0.
\end{align*} \]

Therefore \( H(t) = H(0) \) for all \( t \geq 0 \).
3(c) Assume \( U = \lim_{t \to \infty} u(x,y,z,t) \) is the (constant) steady-state temperature reached by the material in \( D \) at position \((x,y,z)\). Then

\[
H(0) = \lim_{t \to \infty} H(t) = \lim_{t \to \infty} \iiint_D c_p u(x,y,z,t) \, dv = \iiint_D c_p \lim_{t \to \infty} u(x,y,z,t) \, dv
\]

\[
= \iiint_D c_p U \, dv = c_p U \text{ volume}(D). \quad \text{But}
\]

\[
\text{volume}(D) = \iiint_D dv = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{10} r^2 \sin \phi \, dr \, d\phi \, d\theta = \left[ \int_0^{\pi/2} \sin \phi \, d\phi \right] \left[ \int_0^{2\pi} d\theta \right] \left[ \int_0^{10} r^2 \, dr \right]
\]

\[
= \left[ -\cos \phi \right]_0^{\pi/2} \left[ (2\pi)(r^3) \right]_0^{10} = \frac{4\pi}{3} (1000 - 8) = \frac{4\pi}{3} (992).
\]

\[
\text{and}
\]

\[
H(0) = \iiint_D c_p u(x,y,z,0) \, dv = \iiint_D c_p \frac{200}{r \sqrt{x^2 + y^2 + z^2}} \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{10} c_p \frac{200}{\sqrt{r}} \sin \phi \, d\phi \, d\theta \, dr
\]

\[
= \int_0^{2\pi} \int_0^{\pi} c_p \frac{200}{\sqrt{2}} \left[ \sin \phi \right]_0^{\pi/2} \, d\phi \, d\theta = 2\pi c_p (100)(96)(-\cos \phi) \bigg|_0^\pi = 4\pi c_p (9600).
\]

Therefore

\[
U = \frac{H(0)}{c_p \text{ volume}(D)} = \frac{4\pi c_p (9600)}{4\pi c_p (992)} = \frac{28800}{992} = \frac{900}{31} \approx 29.
\]

4. A solution of \( u_t - ku_{xx} = 0 \) in \(-\infty < x < \infty, 0 < t < \infty\), satisfying \( u(x,0) = q(x) \) for all real \( x \) is

\[
u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} q(y) e^{-\frac{(x-y)^2}{4kt}} \, dy \quad (t > 0).
\]

Taking \( k = 1 \) and \( q(y) = y^2 \) for our problem, we find
\[ u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} y e^{-\frac{(x-y)^2}{4t}} \, dy \]  
Make the change of variables \( p = \frac{y-x}{\sqrt{4t}} \). Then \( dp = \frac{dy}{\sqrt{4t}} \), \( y \to +\infty \) implies \( p \to +\infty \), and \( y \to -\infty \) implies \( p \to -\infty \). Therefore,

\[ u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left( x + p \sqrt{4t} \right)^2 e^{-p^2} \, dp = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left( x^2 + 2px\sqrt{4t} + 4t \right) e^{-p^2} \, dp \]

\[ = \frac{x^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \, dp + \frac{2x}{\sqrt{\pi}} \int_{-\infty}^{\infty} pe^{-p^2} \, dp + \frac{4t}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp. \]

But \( \int_{-\infty}^{\infty} e^{-p^2} \, dp = \sqrt{\pi} \), \( \int_{-\infty}^{\infty} pe^{-p^2} \, dp = 0 \) since \( f(p) = pe^{-p^2} \) is an odd function and \( \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp = \frac{\sqrt{\pi}}{2} \).

Consequently \[ u(x, t) = x^2 + 2t \]. Since \( \phi(x) = x^2 \) is not bounded on \(-\infty < x < \infty\), we need to check our answer since technically the formula for \( u = u(x, t) \) is not guaranteed to solve the initial value problem in that case.

But \( u(x, t) = x^2 + 2t \) implies \( u_t = 2 \) and \( u_{xx} = 2 \) so \( u_t - u_{xx} = 2 - 2 = 0 \) for all points \((x, t)\). Also \( u(x, t) = x^2 + 2t \) implies \( u(x, 0) = x^2 \) for all real \( x \). Therefore the boxed function \( u \) above solves the initial value problem.
#5 (Method 1) Let \( u \equiv u(x,t) \) solve (1)-(2)-(3). We take the Fourier transform of both sides of (1) with respect to \( x \) (holding \( t \) fixed):

\[
\frac{\partial^2}{\partial t^2} \hat{u}(s,t) - is \frac{\partial}{\partial t} \hat{u}(s,t) = \hat{f}(s,t) = \hat{f}(s),
\]

(5)

\[
\frac{\partial^2}{\partial t^2} \hat{u}(s,t) + \frac{s^2}{s} \hat{u}(s,t) = \hat{f}(s,t).
\]

The general solution of the second-order, linear, nonhomogeneous ODE (5) in \( t \) (with parameter \( \xi \)) is

\[
\hat{f}(s,t) = U_h(s,t) + U_p(s,t),
\]

where \( U_h(s,t) = c_1(s) \cos(\xi t) + c_2(s) \sin(\xi t) \) is the general solution of the associated homogeneous equation of (5), \( \frac{\partial^2}{\partial t^2} \hat{u}(s,t) + \frac{s^2}{s} \hat{u}(s,t) = 0 \), and \( U_p(s,t) \) is a particular solution of (5). We use variation of parameters to write

\[
U_p(s,t) = v_1(s,t) \cos(\xi t) + v_2(s,t) \sin(\xi t)
\]

where

\[
v_1(s,t) = \int_0^t -Fy_2 \, dt = \int_0^t \frac{\hat{f}(s,\tau) \sin(\xi \tau)}{W(s,\tau)} \, d\tau,
\]

\[
v_2(s,t) = \int_0^t Fy_1 \, dt = \int_0^t \frac{\hat{f}(s,\tau) \cos(\xi \tau)}{W(s,\tau)} \, d\tau,
\]

\( y_1 = \cos(\xi t), \ y_2 = \sin(\xi t) \) form a fundamental set of solutions of (5),

\[
F = \frac{\hat{f}(s,t)}{\xi}, \text{ and }
\]

\[
W = W(s,t) = \begin{vmatrix} \cos(\xi t) & \sin(\xi t) \\ \frac{\partial}{\partial t} \cos(\xi t) & \frac{\partial}{\partial t} \sin(\xi t) \end{vmatrix} = \begin{vmatrix} \cos(\xi t) & \sin(\xi t) \\ -\sin(\xi t) & \cos(\xi t) \end{vmatrix} = \xi
\]

is the Wronskian of \( y_1 \) and \( y_2 \). Thus

\[
\hat{f}(s,t) = c_1(s) \cos(\xi t) + c_2(s) \sin(\xi t) - \cos(\xi t) \int_0^t \frac{\hat{f}(s,\tau) \sin(\xi \tau)}{\xi} \, d\tau + \sin(\xi t) \int_0^t \frac{\hat{f}(s,\tau) \cos(\xi \tau)}{\xi} \, d\tau
\]

(5)

\[
= c_1(s) \cos(\xi t) + c_2(s) \sin(\xi t) + \frac{1}{\xi} \int_0^t \frac{\hat{f}(s,\tau) \sin(\xi (t-\tau))}{\xi} \, d\tau.
\]
Applying the auxiliary condition (2) yields
\[ 0 = \mathcal{F}(u(t)) \bigg|_{t=0} = c_1(t) + 0 + 0 = c_1(t). \]

Next, observe that
\[
\frac{d}{dt} \mathcal{F}(u(t)) = -3c_1(t)\sin(\epsilon t) + 3c_2(t)\cos(\epsilon t) - 3 \int_0^t \frac{\hat{g}(s, t)\sin(\eta t)}{\eta} ds
\]
\[
- \cos(\epsilon t) \cdot \frac{\hat{g}(s, t)\sin(\eta t)}{\eta} + \sin(\epsilon t) \cdot \frac{\hat{g}(s, t)\cos(\eta t)}{\eta}
\]
so by (3) we have
\[ 0 = \mathcal{F}(u(t)) \bigg|_{t=0} = \frac{d}{dt} \mathcal{F}(u(t)) \bigg|_{t=0} = 0 + 3c_2(t) - 0 - 0 + 0 + 0 = 3c_2(t). \]

Thus
\[ \mathcal{F}(u(t)) = \frac{1}{3} \int_0^t \hat{g}(s, \tau)\sin(\eta (t-\tau)) d\tau. \] (4)

Using formula A in the table of Fourier transforms with \( b = t-\tau \), we see that
\[ \frac{\sin(\eta (t-\tau))}{\eta} = \sqrt{\frac{\pi}{2}} \mathcal{F}(\chi_{(-t, t)}(x)). \]

Also, since \( \hat{g}(x,t) = \sqrt{\frac{\pi}{2}} \chi_{(-t,t)}(x) \),
\[ \hat{g}(s, \tau) = \sqrt{\frac{\pi}{2}} \mathcal{F}(\chi_{(-t, t)}(x)). \]

Substituting in (4) and using the convolution formula \( \mathcal{F}(g \ast h)(s) = \sqrt{2\pi} \mathcal{F}g(s) \mathcal{F}h(s) \) leads to
\[ \mathcal{F}(u(t)) = \int_0^t \sqrt{\frac{\pi}{2}} \mathcal{F}(\chi_{(-\tau, \tau)}(x)) \sqrt{\frac{\pi}{2}} \mathcal{F}(\chi_{(-t, t-\tau)}(x)) d\tau \]
\[
= \frac{\pi}{2} \int_0^t \frac{1}{\sqrt{2\pi}} \mathcal{F}(\chi_{(-\tau, \tau)} \ast \chi_{(-t, t-\tau)}(x)) d\tau
= \mathcal{F} \left( \frac{\pi}{2\sqrt{2\pi}} \int_0^t \chi_{(-\tau, \tau)} \chi_{(-t, t-\tau)} d\tau \right). \]
The inversion theorem then implies

\[ u(x,t) = \frac{\sqrt{\pi}}{2\sqrt{2}} \int_0^t \left( \int_{-\infty}^\infty x_{(-\tau,\tau)}(-y) X_{(-\tau,\tau), t-	au}(y) \right) dy \, d\tau. \]

\[ \therefore \quad u(x,t) = \sqrt{\pi} \int_0^t \left( \int_0^t \mathcal{X}_{(-\tau,\tau)}(s) X_{(-t\tau, t\tau), x+t\tau}(s) \right) ds \, d\tau \]
\[ = \sqrt{\pi} \int_0^t \left( \int_0^t \text{length of } \{ x(t\tau, t\tau) \cap (x(t\tau, t\tau) + t\tau) \} \right) ds \, d\tau \]
\[ = \sqrt{\pi} \left[ \int_0^t 2\tau \, d\tau + \int_{t+\tau}^t (t-x) \, d\tau + \int_t^{t+\tau} 2(t-\tau) \, d\tau \right] \mathcal{X}_{(-\tau,\tau)}(x). \]

\[ \therefore \quad \boxed{ u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}} \left( t^2 - x^2 \right) \mathcal{X}_{(-\tau,\tau)}(x) } \]
if \(-\infty < x < \infty\) and \(0 < t < \infty\).

Since \( f(x,-t) = f(x,t) \) and the pde \( u_{tt} - u_{xx} = f(x,t) \) is invariant under reflection through the x-axis, \((x,t)\mapsto(x,-t)\), it follows that

\[ \boxed{ u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}} \left( t^2 - x^2 \right) \mathcal{X}_{(-\tau,\tau)}(x) } \]

solves (1) - (2) - (3) in the entire xt-plane.

Note: This is a generalized solution to (1) - (2) - (3) in the xt-plane. In particular,

\[ u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}} \left( t^2 - x^2 \right) \mathcal{X}_{(-\tau,\tau)}(x) \]

is a continuous function in the xt-plane such that

\[ u_{tt}(x,t) - u_{xx}(x,t) = \begin{cases} \frac{\sqrt{\pi}}{4\sqrt{2}} & \text{if } |x| < t^2 \\ 0 & \text{if } |x| > t^2 \end{cases} = f(x,t) \text{ if } |x| < t^2 \]

and \( u(x,0) = 0 = u_t(x,0) \) if \(-\infty < x < \infty\).
\[ \hat{u}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) \sin(\xi(t-r)) \, d\xi. \]

But \( f(x,t) = \sqrt{2\pi} \mathcal{F}_x(x), \) so formula A in the Fourier transform table implies
\[ \hat{f}(\xi,t) = \frac{\sin(\frac{\xi t}{2})}{\frac{\xi}{2}}. \]
Substituting in (7) gives
\[ \hat{f}(u)(\xi) = \frac{1}{\sqrt{\pi}} \int_{0}^{t} \sin(\frac{\xi r}{2}) \sin(\frac{\xi(t-r)}{2}) \, dr. \]

Using the identity \( \sin A \sin B = \frac{1}{2} \left[ \cos(A-B) - \cos(A+B) \right] \) gives
\[
\begin{align*}
\hat{f}(u)(\xi) &= \frac{1}{2\sqrt{\pi}} \int_{0}^{t} \left[ \cos(\frac{3\xi t}{2} - \frac{3\xi t}{2}) - \cos(\frac{3\xi t}{2}) \right] \, dr \\
&= \frac{1}{2\sqrt{\pi}} \left[ \frac{\sin(\frac{3\xi t}{2})}{\frac{3\xi}{2}} - t \cos(\frac{3\xi t}{2}) \right]_{\tau=0}^{t} \\
&= \frac{1}{2\sqrt{\pi}} \left[ \frac{\sin(\frac{3\xi t}{2})}{\frac{3\xi}{2}} - t \cos(\frac{3\xi t}{2}) \right].
\end{align*}
\]

Taking the inverse transform yields
\[
\begin{align*}
u(x,t) &= \lim_{M \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-M}^{M} \hat{f}(u)(\xi) e^{ix\xi} \, d\xi \\
&= \lim_{M \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-M}^{M} \frac{1}{2\sqrt{\pi}} \left[ \frac{\sin(\frac{3\xi t}{2})}{\frac{3\xi}{2}} - t \cos(\frac{3\xi t}{2}) \right] \cos(\xi x) \, d\xi \\
&= \lim_{M \to \infty} \frac{2}{\sqrt{2\pi}} \int_{0}^{M} \left( \frac{\sin(\xi t)}{2\xi^3} - \frac{t \cos(\xi t)}{2\xi^2} \right) \cos(\xi x) \, d\xi.
\end{align*}
\]

Using the identities \( \sin A \cos B = \frac{1}{2} \left[ \sin(A+B) + \sin(A-B) \right] \) and \( \cos A \cos B = \frac{1}{2} \left[ \cos(A+B) + \cos(A+B) \right] \) yields
\[ u(x,t) = \lim_{M \to \infty} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \left[ \frac{1}{4\pi^2} \left( \sin(\pi(x+\xi)) + \sin(\pi(t-\xi)) \right) - \frac{\pi}{4\pi^2} \left( \cos(\pi(x+\xi)) + \cos(\pi(t-\xi)) \right) \right] d\xi \\
\]
\[ = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{\sin(\pi(t-x)) - \pi(t-x) \cos(\pi(t-x))}{4\pi^3} \, d\xi + \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{\sin(\pi(t+x)) - \pi(t+x) \cos(\pi(t+x))}{4\pi^3} \, d\xi \]
\[ + \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{3 \pi \cos(\pi(t+x)) - 3 \pi \cos(\pi(t-x))}{4\pi^3} \, d\xi \]
\[ = \frac{(t-x)^2}{2\sqrt{2\pi}} \int_{0}^{\infty} \frac{\sin(\pi(t-x)) - \pi(t-x) \cos(\pi(t-x))}{4\pi^2} \, d\xi + \frac{(t+x)^2}{2\sqrt{2\pi}} \int_{0}^{\infty} \frac{\sin(\pi(t+x)) - \pi(t+x) \cos(\pi(t+x))}{4\pi^2} \, d\xi \]
\[ = \frac{x}{2\sqrt{2\pi}} \int_{0}^{\infty} \frac{\cos(\pi(t+x)) - \cos(\pi(t-x))}{\pi^2} \, d\xi \]

Using the identities \[ \int_{0}^{\infty} \frac{\sin(\lambda) - \lambda \cos(\lambda)}{\lambda^3} \, d\lambda = \frac{\pi}{4} \] (See Gradshteyn & Ryzhik 3.784(7))
and \[ \int_{0}^{\infty} \frac{\cos(\lambda x) - \cos(\lambda y)}{\lambda^2} \, d\lambda = \frac{(b-a)\pi}{2} \] (See Gradshteyn & Ryzhik 3.784(3),
leads to
\[ u(x,t) = \frac{(t-x)^2}{2\sqrt{2\pi}} \cdot \frac{\pi}{4} + \frac{(t+x)^2}{2\sqrt{2\pi}} \cdot \frac{\pi}{4} \]
\[ + \frac{x}{2\sqrt{2\pi}} \cdot \frac{(t-x)\pi}{2} \]
\[ = \frac{\sqrt{\pi}}{8\sqrt{2}} \left[ t^2 - 2tx + x^2 + t^2 + 2tx + x^2 - 4x^2 \right] \]
\[ = \frac{\sqrt{\pi}}{4\sqrt{2}} (t^2 - x^2). \quad \text{(Valid if } |x| < t \text{ and } t > 0.) \]

Proceeding as in method 1, it follows that
\[ u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}} (t^2 - x^2) \chi(-111,111)(x) \]
in the xt-plane.
#6. (a) Since \( f \) is an even function, the full Fourier series on \((-\pi, \pi)\):
\[
f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(nx) + b_n \sin(nx) \right]
\]
is actually a cosine series since
\[
b_n = \frac{\langle f, \sin(nx) \rangle}{\langle \sin(x), \sin(x) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x) \sin(nx) \, dx}{\sin(x)} = 0 \quad (n = 1, 3, 5, \ldots).
\]
We also have
\[
a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \frac{3}{3\pi} \int_{0}^{\pi} = \frac{\pi^2}{3}
\]
and if \( n \geq 1 \),
\[
a_n = \frac{\langle f, \cos(nx) \rangle}{\langle \cos(x), \cos(x) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x) \cos(nx) \, dx}{\cos(x)} = \frac{2}{\pi} \int_{0}^{\pi} \frac{x \cos(nx) \, dx}{\cos(x)} = \frac{2}{\pi} \int_{0}^{\pi} \frac{x \sin(nx) \, dx}{\sin(x)} - \frac{2}{\pi} \int_{0}^{\pi} \frac{x \sin(nx) \, dx}{\sin(x)}
\]
\[
= -\frac{4}{\pi n} \left[ \frac{\sin(nx)}{n} \right]_{0}^{\pi} = \frac{4}{\pi n^2} f(\pi) = \frac{4}{\pi n^2} \frac{\pi^2}{3} = \frac{4}{\pi n^2} \frac{\pi^2}{3}.
\]
Therefore the full Fourier series of \( f \) is
\[
\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi n^2} \cos(nx).
\]

(b) It is clear that \( f \) is continuous and \( f' \) is piecewise continuous on the real line. Since \( f \) is \( 2\pi \)-periodic, Theorem 4.9\(^a\) from the Fourier series convergence theorems guarantees that the full Fourier series of \( f \) converges to \( f(x^+) + f(x^-) \) for every real \( x \). But \( f(x^+) = f(x) \) and \( f(x^-) = f(x) \) for every real \( x \) since \( f \) is continuous on the real line. Therefore the full Fourier series of \( f \) converges to \( f(x) \) for all real \( x \). In particular,
\[
x^2 = f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi n^2} \cos(nx)
\]
for all \(-\pi \leq x \leq \pi\).
\[ H. (c) \quad \text{Taking } x = 0 \text{ in the identity from part (b) gives} \]
\[
0 = \frac{\pi}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \quad \text{so} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12} .
\]

\[ \text{Taking } x = \pi \text{ in the identity from part (b) leads to} \]
\[
\pi^2 = \frac{\pi}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\pi)}{n^2} \quad \text{so} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \approx \frac{1}{4} \left( \frac{\pi^2}{3} \right) = \frac{\pi^2}{6} .
\]

\[ H. (a) \quad \text{We seek nontrivial solutions of the form } u(x,t) = \Xi(x)T(t) \text{ to (1)-(2)-(3)-(4).} \]

Substituting in (1) yields \[ \Xi'(x)T''(t) - \Xi''(x)T(t) + 2\Xi(x)T(t) = 0 \quad \text{so} \quad (\ast) \]
\[
\frac{-\Xi''(x)}{\Xi(x)} = -\left( \frac{T''(t) + 2T'(t)}{T(t)} \right) = \text{constant} = \lambda .
\]

Substituting \( u(x,t) = \Xi(x)T(t) \) in (2) and (3) gives \[ \Xi'(x)T(t) = 0 = \Xi'(x)T(t) \] for \( t \geq 0 \)
and in (4) gives \[ \Xi(x)T(0) = 0 \] for \( 0 \leq x \leq \pi \). Thus nontriviality of \( u \) and \( (\ast) \) imply
\[ \Xi''(x) + \lambda \Xi(x) = 0, \quad \Xi'(x) = 0 = \Xi'(x), \]
\[ T''(t) + 2T'(t) + \lambda T(t) = 0, \quad T(0) = 0 . \]

By Sec. 4.2, the eigenvalues and eigenfunctions of (6) are \( \lambda_n = n^2 \) and \( \Xi_n(x) = \cos(nx) \)
where \( n = 0, 1, 2, \ldots \). Substituting \( \lambda = \lambda_n = n^2 \) in (6) yields
\[ T''_n(t) + 2T'_n(t) + n^2T_n(t) = 0, \quad T_n(0) = 0 . \]

We look for solutions to the differential equation in (8) of the form \( T_n(t) = e^{r_n t} \) where \( r_n \) is a constant. Substituting \( T_n(t) = e^{r_n t} \) in (8) leads to
\[
r_n^2 + 2r_n + n^2 = 0 \quad \text{so} \quad r_n = -1 \pm \sqrt{1-n^2} .
\]

If \( n = 0 \) then \( r_0 = 0 \) or \( r_0 = -2 \) so \( T_0(t) = c_1 + c_2 e^{-2t} \) is the general solution of
the DE in (8).

If \( n = 1 \) then \( r_1 = -1 \) with multiplicity two so \( T_1(t) = c_1 e^t + c_2 t e^t \) is the
general solution of the DE in (8).
If \( n \geq 2 \) then \( r_n = -1 \pm i\sqrt{n^2 - 1} \) so \( T_n(t) = e^{-t} (c_1 \cos(\sqrt{n^2 - 1} t) + c_2 \sin(\sqrt{n^2 - 1} t)) \) is the general solution of the DE in (3). Taking into account the initial condition in (3), we have, up to a constant factor,

\[
T_0(t) = 1 - e^{-2t}, \quad T_1(t) = te^{-t}, \quad \text{and} \quad T_n(t) = e^{-t} \sin(\sqrt{n^2 - 1} t) \quad \text{if} \quad n \geq 2.
\]

By the superposition principle, a formal solution to (1)-(2)-(3)-(4) is

\[
u(x,t) = \sum_{n=0}^{\infty} A_n \varphi_n(x) T_n(t) = A_0 (1 - e^{-2t}) + A_1 \cos(t) e^{-t} + \sum_{n=2}^{\infty} A_n \cos(nx) e^{-t} \left[ \sqrt{n^2 - 1} \cos(\sqrt{n^2 - 1} t) - \sin(\sqrt{n^2 - 1} t) \right]
\]

where \( A_0, A_1, A_2, \ldots \) are arbitrary constants. Then

\[
u_t(x,t) = 2A_0 e^{-2t} + A_1 \cos(t) (-e^{-t}) + \sum_{n=2}^{\infty} A_n \cos(nx) e^{-t} \left[ \sqrt{n^2 - 1} \cos(\sqrt{n^2 - 1} t) - \sin(\sqrt{n^2 - 1} t) \right]
\]

so to satisfy (5) we must have

\[
x^2 = \nu_t(x,0) = 2A_0 + A_1 \cos(x) + \sum_{n=2}^{\infty} A_n \sqrt{n^2 - 1} \cos(nx) \quad \text{if} \quad 0 \leq x \leq \pi.
\]

By problem 6,

\[
2A_0 = \frac{\pi^2}{3}, \quad A_1 = -4, \quad \text{and for} \quad n \geq 2, \quad A_n \sqrt{n^2 - 1} = \frac{(-1)^n}{n^2}.
\]

Thus a solution to (1)-(2)-(3)-(4)-(5) is

\[
u(x,t) = \frac{\pi^2}{6} (1 - e^{-2t}) - 4e^{-t} \cos(x) + 4e^{-t} \sum_{n=2}^{\infty} \frac{(-1)^n \cos(nx) \sin(\sqrt{n^2 - 1} t)}{n^2 \sqrt{n^2 - 1}}
\]

(b) Let \( u = u(x,t) \) satisfy (1)-(2)-(3) and consider \( E(t) = \frac{1}{2} \int_0^\pi \left[ u_x^2(x,t) + u_t^2(x,t) \right] dx \), the energy function for \( u \) on the interval \( 0 \leq t < \infty \). We have

\[
E(t) = \frac{1}{2} \int_0^\pi \frac{2}{dx} \left[ u_x^2(x,t) + u_t^2(x,t) \right] dx
\]

where

\[
= \int_0^\pi \left[ u_t(x,t) u_{tt}(x,t) + u_x(x,t) u_{xt}(x,t) \right] dx
\]

\[
= \int_0^\pi u_t(x,t) u_{tt}(x,t) dx + u_x(x,t) u_{xt}(x,t) \bigg|_{x=0}^{x=\pi} - \int_0^\pi u_t(x,t) u_{xx}(x,t) dx
\]

by (2)-(3).
\[ E'(t) = \int_0^\pi u_t(x,t)[u_{tt}(x,t) - u_{xx}(x,t)]dx \]
\[ = -2\int_0^\pi u_x^2(x,t)dx \quad \text{(by (1))} \]
\[ \leq 0. \]

Consequently \( E = E(t) \) is decreasing on \( t \geq 0 \).

(c) Yes, there is only one solution to the problem in part (a). For suppose that \( u = v(x,t) \) is another solution to this problem and consider \( w(x,t) = u(x,t) - v(x,t) \) for \( 0 \leq x \leq \pi \) and \( t \geq 0 \). Then \( w \) is continuous and solves

\[ (1') \quad w_{tt} - w_{xx} + 2w_t = 0 \quad \text{if} \quad 0 < x < \pi, 0 < t < \infty, \]
\[ (2')- (3') \quad w_x(0,t) = 0 = w_x(\pi,t) \quad \text{if} \quad t \geq 0, \]
\[ (4') - (5') \quad w(x,0) = 0 = w_t(x,0) \quad \text{if} \quad 0 \leq x \leq \pi. \]

By part (b), the energy function \( E(t) = \frac{1}{2} \int_0^\pi [w_t^2(x,t) + w_x^2(x,t)]dx \) is decreasing on \( 0 \leq t < \infty \). But then by (5') and differentiation of (4'),

\[ 0 \leq E(t) \leq E(0) = \frac{1}{2} \int_0^\pi [w_t^2(x,0) + w_x^2(x,0)]dx = 0. \]

Consequently the vanishing theorem implies \( w_t(x,t) = 0 = w_x(x,t) \) if \( 0 \leq x \leq \pi \) and \( 0 \leq t < \infty \), and hence \( w(x,t) = \text{constant in the strip} \ 0 \leq x \leq \pi, 0 \leq t < \infty. \)

But (4') implies the constant is zero; i.e. \( u(x,t) - v(x,t) = w(x,t) = 0 \) for all \( 0 \leq x \leq \pi \) and \( 0 \leq t < \infty. \) Since \( v(x,t) = u(x,t) \) in the strip \( 0 \leq x \leq \pi, 0 \leq t < \infty, \) there is only one solution to the problem in (a).
Math 325  
Summer 2012  
Final Exam

\[ n = 16 \]

\[ \text{mean} = 107.9 \]

\[ \text{median} = 104 \]

\[ \text{standard deviation} = 53.0 \]

Distribution of Scores

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