Mathematics 325 | Final Exam Summer 2010 | Name: Dr. Grow

1.(25 pts.) Consider the partial differential equation

(*) $$\sqrt{1+y^2}u_y + 2xyu_x = 0.$$

(a) State the order and type (linear homogeneous, linear inhomogeneous, nonlinear) of (*).
(b) Find and sketch graphs of three characteristic curves of (*).
(c) Find the general solution of (*) in the xy – plane.
(d) What is the solution to (*) satisfying $u(x,0) = x^3$ for $-\infty < x < \infty$?

2.(25 pts.) Consider the partial differential equation

(+) $$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$

(a) State the order and type (linear homogeneous, linear inhomogeneous, nonlinear, hyperbolic, elliptic, etc.) of (+).
(b) Find the general solution of (+) in the xy – plane.
(c) Find the solution of (+) that satisfies $u(x,0) = 4x^3$ and $u_y(x,0) = -4x^2$ for $-\infty < x < \infty$.

3.(25 pts.) Find the solution of $u_t - u_{xx} = 0$ in the upper half-plane satisfying $u(x,0) = e^{-x^2}$ for $-\infty < x < \infty$.

4.(25 pts.) Use Fourier transform methods to derive the formula

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+t-\tau} f(s,\tau)\,ds\,d\tau$$

for the solution to the wave equation with source in the xt – plane,

$$u_{tt} - u_{xx} = f(x,t),$$

satisfying $u(x,0) = 0 = u_t(x,0)$ for $-\infty < x < \infty$.

5.(25 pts.) (a) Show that the Fourier cosine series for $f(x) = x^2$ on $[0,\pi]$ is

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos(nx)}{n^2}.$$

(b) Discuss the pointwise convergence or divergence of the Fourier cosine series of f on the interval $[0,\pi]$.
(c) Show that the Fourier cosine series of f converges uniformly to f on $[0,\pi]$.
(d) Use the results above to help find the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

6.(25 pts.) (a) Find a solution to

(1) $$u_{tt} - u_{xx} + 2u_t = 0 \text{ in } 0 < x < \pi,\ 0 < t < \infty,$$

satisfying

(2)-(3) $$u_x(0,t) = 0 = u_x(\pi,t) \text{ for } 0 \le t < \infty,$$

(4) $$u(x,0) = 0 \text{ for } 0 \le x \le \pi,$$

(5) $$u_t(x,0) = x^2 \text{ for } 0 \le x \le \pi.$$

(b) Show that if $u = u(x,t)$ satisfies (1)-(2)-(3) then its energy function

$$E(t)=\frac{1}{2}\int_0^{\pi}\left[u_t^{\,2}(x,t)+u_x^{\,2}(x,t)\right]dx$$

is decreasing on $0\le t<\infty$.

(c) Is there at most one solution to the problem in part (a)? Justify your answer.

7.(25 pts.) The material in a thin circular disk of unit radius has reached a steady-state temperature distribution. The material is held at 50 degrees Centigrade on the top half of the disk's edge and at -50 degrees Centigrade on the bottom half of its edge.

(a) Write the partial differential equation and boundary condition(s) governing the temperature function of the material.

(b) Write a formula for the temperature function of the material.

(c) What is the temperature of the material at the center of the disk? Justify your answer.

(d) What is the temperature of the material at a general point in the disk? [You must evaluate any integral expressions for credit on this part of the problem.]

8.(25 pts.) (a) Solve the problem of heat conduction in a square laminar region,

$$u_t-\left(u_{xx}+u_{yy}\right)=0 \text{ for } 0<x<\pi,\ 0<y<\pi,\ 0<t<\infty,$$

given that for all times $t\ge 0$, the four edges of the square are insulated and initially the temperature function satisfies

$$u(x,y,0)=\cos^2(x)\cos^2(y) \text{ for } 0\le x\le\pi,\ 0\le y\le\pi.$$

(b) Is there at most one solution to the problem in (a)? Justify your answer.

(c) What is the steady-state temperature distribution in the square? That is, what is

$$U(x,y)=\lim_{t\to\infty}u(x,y,t) \text{ for } 0\le x\le\pi,\ 0\le y\le\pi?$$

$\boxed{\#1}$ (a) $\sqrt{1+y^2}\,u_y + 2xy\,u_x \overset{(*)}{=} 0$ is a $\boxed{\text{first order, linear, homogeneous p.d.e.}}$

(b) The characteristics of $a(x,y)u_x + b(x,y)u_y = 0$ are curves satisfying

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

Therefore, the characteristics are given by

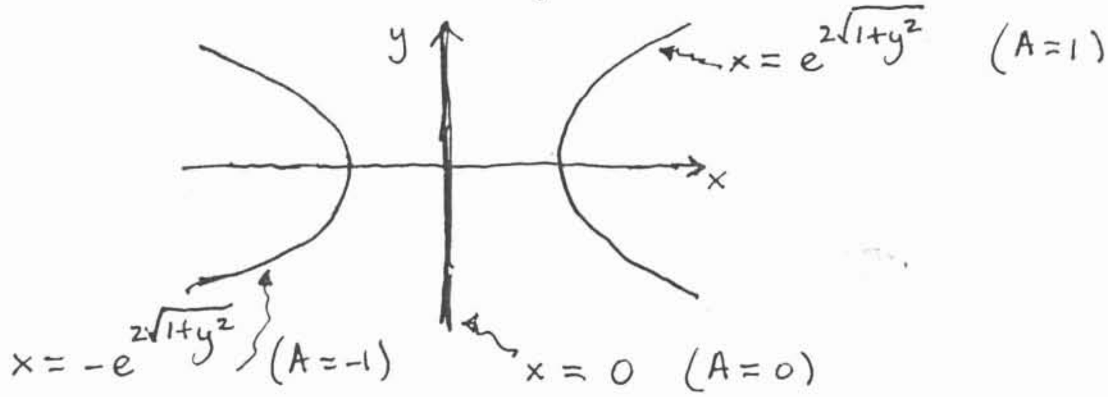
$$\frac{dy}{dx} = \frac{\sqrt{1+y^2}}{2xy}$$

for our p.d.e. Separating variables and integrating gives characteristics:

$$2\sqrt{1+y^2} + c = \int \frac{2y\,dy}{\sqrt{1+y^2}} = \int \frac{dx}{x} = \ln|x| \Rightarrow \boxed{Ae^{2\sqrt{1+y^2}} = x}$$

where A is an arbitrary constant.

10 pts.



(c) Along a characteristic curve, the solution to the p.d.e. is constant. Thus along such curves we have $u(x,y) = u\left(Ae^{2\sqrt{1+y^2}}, y\right) = u\left(Ae^2, 0\right) = f(A)$. Thus the general solution of the p.d.e. is $\boxed{u(x,y) = f\left(xe^{-2\sqrt{1+y^2}}\right)}$ where f is any C^1-function of a single real variable.

5 pts.

(d) A particular solution to the p.d.e. satisfying $u(x,0) = x^3$ for all real x means $x^3 = f(xe^{-2})$ for $-\infty < x < \infty$. But then

$e^6(xe^{-2})^3 = f(xe^{-2})$ for $-\infty < x < \infty$, so $f(s) = e^6 s^3$ for $-\infty < s < \infty$

Consequently, $u(x,y) = e^6\left(xe^{-2\sqrt{1+y^2}}\right)^3 = \boxed{x^3 e^{6(1-\sqrt{1+y^2})}}$. 5 pts

4 pts

$\boxed{\#2}$ (a) The p.d.e. $(+)$ $u_{xx} - 4u_{xy} + 4u_{yy} = 0$ is $\boxed{\text{second order, linear, homogeneous, and parabolic}}$ because $B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$.

(b) Note that $(+)$ can be expressed as $\left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} + 4\frac{\partial^2}{\partial y^2}\right)u = 0$, or equivalently $\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)^2 u = 0$. This suggests the change-of-coordinates $\xi = x - 2y$, $\eta = 2x + y$. By the chain rule we then have

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\cdot\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\cdot\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}$$

and
$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = -2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Therefore $\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta} - 2\left(-2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = 5\frac{\partial}{\partial \xi}$ so $(+)$ is equivalent to $\left(5\frac{\partial}{\partial \xi}\right)^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial \xi^2} = 0 \Rightarrow \frac{\partial u}{\partial \xi} = c_1(\eta) \Rightarrow u = \xi c_1(\eta) + c_2(\eta)$

That is,
$$\boxed{u(x,y) = (x-2y)f(2x+y) + g(2x+y)}$$ 13 pts

where f and g are C^2-functions of a single real variable.

(c) A particular solution of $(+)$ satisfying $u(x,0) = 4x^3$ and $u_y(x,0) = -4x^2$ for all real x leads to the system

① $\quad xf(2x) + g(2x) = 4x^3$

② $\quad -2f(2x) + xf'(2x) + g'(2x) = -4x^2$

for all real x. Differentiating ① leads to the system

①' $\quad f(2x) + 2xf'(2x) + 2g'(2x) = 12x^2$

② $\quad -2f(2x) + xf'(2x) + g'(2x) = -4x^2$.

Multiplying ② by 2 and subtracting the result from ①' yields

$$5f(2x) = 20x^2 \quad \Rightarrow \quad f(2x) = 4x^2 = (2x)^2.$$

Therefore $f(s) = s^2$ for all real s. Substituting this in ① we have

$$x(2x)^2 + g(2x) = 4x^3 \quad \Rightarrow \quad g(2x) = 0.$$

That is, $g(s) = 0$ for all real s. Consequently the particular solution is

$$u(x,y) = (x-2y)f(2x+y) + g(2x+y)$$
$$= \boxed{(x-2y)(2x+y)^2} \qquad \text{8 pts.}$$

or equivalently,

$$\boxed{u(x,y) = 4x^3 - 4x^2y - 7xy^2 - 2y^3}.$$

$\boxed{\#3}$ Recall that $u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \varphi(y) e^{-\frac{(x-y)^2}{4kt}} \, dy$ solves

$u_t - k u_{xx} = 0$ in the upper half-plane, subject to $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$.

In our case $k=1$ and $\varphi(x) = e^{-x^2}$ so

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2} \cdot e^{-\frac{(x-y)^2}{4t}} \, dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-4ty^2-(x-y)^2}{4t}} \, dy.$$

5 pts to here

We complete the square in y in the exponent of the integrand:

$$\begin{aligned}
\frac{-4ty^2-(x-y)^2}{4t} &= \frac{-4ty^2 - x^2 + 2xy - y^2}{4t} \\
&= \frac{-\left[(1+4t)y^2 - 2xy + \frac{x^2}{1+4t}\right] + \frac{x^2}{1+4t} - x^2}{4t} \\
&= \frac{-\left(\sqrt{1+4t}\, y - \frac{x}{\sqrt{1+4t}}\right)^2 - \frac{4tx^2}{1+4t}}{4t} \\
&= \frac{-\left(\sqrt{1+4t}\, y - \frac{x}{\sqrt{1+4t}}\right)^2}{4t} - \frac{x^2}{1+4t}.
\end{aligned}$$

15 pts to here

Therefore

$$u(x,t) = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-\left(\sqrt{1+4t}\, y - \frac{x}{\sqrt{1+4t}}\right)^2}{4t}} \, dy.$$

20 pts to here

Let $p = \frac{\sqrt{1+4t}\, y - \frac{x}{\sqrt{1+4t}}}{\sqrt{4t}}$. Then $dp = \sqrt{\frac{1+4t}{4t}}\, dy$ so

$$u(x,t) = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{4\pi t}} \cdot \sqrt{\frac{4t}{1+4t}} \int_{-\infty}^{\infty} e^{-p^2}\,dp = \boxed{\frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}}\ .$$

25 pts to here.

$\boxed{\#4}$ Let $u = u(x,t)$ be a solution to

$$① \qquad u_{tt} - u_{xx} = f(x,t) \quad \text{in } -\infty < x < \infty,\ -\infty < t < \infty,$$

satisfying

$$②\text{-}③ \qquad u(x,0) = 0 = u_t(x,0) \quad \text{for } -\infty < x < \infty.$$

Taking the Fourier transform of both sides of ① with respect to x yields

$$\mathcal{F}(u_{tt})(\xi) - \mathcal{F}(u_{xx})(\xi) = \mathcal{F}(f(\cdot,t))(\xi)$$

$$\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$$

$$(*) \qquad \frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) + \xi^2\mathcal{F}(u)(\xi) = \hat{f}(\xi,t)\ .$$

The general solution of this second order ordinary differential equation in the independent variable t (with parameter ξ) is

$$\mathcal{F}(u)(\xi) = U_h(\xi,t) + U_p(\xi,t)$$

where $U_h(\xi,t) = c_1(\xi)\cos(\xi t) + c_2(\xi)\sin(\xi t)$ is the general solution of $\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) + \xi^2\mathcal{F}(u)(\xi) = 0$ and a particular solution of $(*)$ is given by variation of parameters: $U_p(\xi,t) = u_1(\xi,t)\cos(\xi t) + u_2(\xi,t)\sin(\xi t)$ with

15 pts to here.

(5 pts)

$$u_1(\xi,t) = \int_0^t \frac{-F y_2}{W}\,d\tau = \int_0^t \frac{-\hat{f}(\xi,\tau)\sin(\xi\tau)}{\xi}\,d\tau\ ,$$

and $$u_2(\xi,t) = \int_0^t \frac{F y_1}{W}\,d\tau = \int_0^t \frac{\hat{f}(\xi,\tau)\cos(\xi\tau)}{\xi}\,d\tau\ .$$

$$W = \text{Wronskian of } y_1 \text{ and } y_2 = \begin{vmatrix} \cos(\xi\tau) & \sin(\xi\tau) \\ -\xi\sin(\xi\tau) & \xi\cos(\xi\tau) \end{vmatrix} = \xi\cos^2(\xi\tau) + \xi\sin^2(\xi\tau) = \xi$$

Therefore $\mathcal{F}(u)(\xi) = c_1(\xi)\cos(\xi t) + c_2(\xi)\sin(\xi t)$

$$+ \cos(\xi t)\int_0^t -\frac{\hat{f}(\xi,\tau)\sin(\xi\tau)}{\xi}\,d\tau + \sin(\xi t)\int_0^t \frac{\hat{f}(\xi,\tau)\cos(\xi\tau)}{\xi}\,d\tau$$

$$= c_1(\xi)\cos(\xi t) + c_2(\xi)\sin(\xi t) + \int_0^t \frac{\hat{f}(\xi,\tau)}{\xi}\left[\sin(\xi t)\cos(\xi\tau) - \cos(\xi t)\sin(\xi\tau)\right]d\tau$$

$$= c_1(\xi)\cos(\xi t) + c_2(\xi)\sin(\xi t) + \int_0^t \frac{\hat{f}(\xi,\tau)\sin(\xi(t-\tau))}{\xi}\,d\tau .$$

We use ② and ③ to identify $c_1(\xi)$ and $c_2(\xi)$. First, ② implies

$$0 = \mathcal{F}(u(\cdot,0))(\xi) = \left(c_1(\xi)\cos(\xi t) + c_2(\xi)\sin(\xi t) + \int_0^t \frac{\hat{f}(\xi,\tau)\sin(\xi(t-\tau))}{\xi}\,d\tau\right)\Bigg|_{t=0}$$

$$= c_1(\xi) .$$

17 pts to here

Next, we apply ③. We will need the Leibniz differentiation rule:

$$(*) \qquad \frac{d}{dt}\left(\int_0^t F(\tau,t)\,d\tau\right) = \int_0^t \frac{\partial F}{\partial t}(\tau,t)\,d\tau + F(t,t).$$

By ③ and (*),

$$0 = \mathcal{F}(u_t(\cdot,0))(\xi) = \frac{\partial}{\partial t}\mathcal{F}(u)(\xi)\Big|_{t=0}$$

$$= \Bigg[-\xi c_1(\xi)\sin(\xi t) + \xi c_2(\xi)\cos(\xi t) + \int_0^t \frac{\partial}{\partial t}\frac{\hat{f}(\xi,\tau)\sin(\xi(t-\tau))}{\xi}\,d\tau + \frac{\hat{f}(\xi,t)\sin(\xi(t-t))}{\xi}\Bigg]\Bigg|_{t=0}$$

$$= \xi c_2(\xi) .$$

19 pts to here

, since $c_1(\xi) = 0 = c_2(\xi)$

Consequently, $\mathcal{F}(u)(\xi) = \displaystyle\int_0^t \frac{\hat{f}(\xi,\tau)\sin(\xi(t-\tau))}{\xi}\,d\tau$. 2 pts. to here

We will need two properties of Fourier transforms to finish:

(A) If $g_a(x) = g(x-a)$ then $\mathcal{F}(g_a)(\xi) = e^{-i\xi a}\hat{g}(\xi)$.

(B) If $H(x) = \displaystyle\int_0^x h(s)\,ds$ then $\hat{H}(\xi) = \dfrac{\hat{h}(\xi)}{i\xi}$.

From (B) and $F(x,\tau) = \displaystyle\int_0^x f(s,\tau)\,ds$, it follows that $\dfrac{\hat{f}(\xi,\tau)}{\xi} = i\,\widehat{F(\xi,\tau)}$.

From (A) it follows that $\sin(\xi a)\hat{g}(\xi) = \frac{1}{2i}\left(e^{i\xi a} - e^{-i\xi a}\right)\hat{g}(\xi) = \frac{1}{2i}(g_{-a} - g_a)(\xi)$

$= \frac{1}{2i}\mathcal{F}\big(g(\cdot+a) - g(\cdot-a)\big)$

Thus

$$\mathcal{F}(u)(\xi) = \int_0^t i\,\hat{F}(\xi,\tau)\sin(\xi(t-\tau))\,d\tau$$

$$= \int_0^t i\cdot\frac{1}{2i}\mathcal{F}\Big(F(\cdot+(t-\tau),\tau) - F(\cdot-(t-\tau),\tau)\Big)(\xi)\,d\tau$$

$$= \mathcal{F}\left(\frac{1}{2}\int_0^t \Big[F(\cdot+t-\tau,\tau) - F(\cdot-(t-\tau),\tau)\Big]d\tau\right)(\xi)$$

where we have switched the orders of integration in the last two lines.

By the uniqueness theorem for Fourier transforms, it follows that

$$u(x,t) = \frac{1}{2}\int_0^t\Big[F(x+t-\tau,\tau) - F(x-(t-\tau),\tau)\Big]d\tau$$

$$= \frac{1}{2}\int_0^t\left[\int_0^{x+t-\tau} f(s,\tau)\,ds - \int_0^{x-(t-\tau)} f(s,\tau)\,ds\right]d\tau$$

$$= \boxed{\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+t-\tau} f(s,\tau)\,ds\,d\tau}.$$

25 pts. to here.

$\boxed{\#5}$ (a) The Fourier coefficients of $f(x) = x^2$ on $[0,\pi]$ with respect to the orthogonal set of functions $\Phi = \left\{ \cos(n\cdot) \right\}_{n=0}^{\infty}$ on $[0,\pi]$ are:

$$a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \left.\frac{x^3}{3\pi}\right|_0^{\pi} = \frac{\pi^2}{3}$$

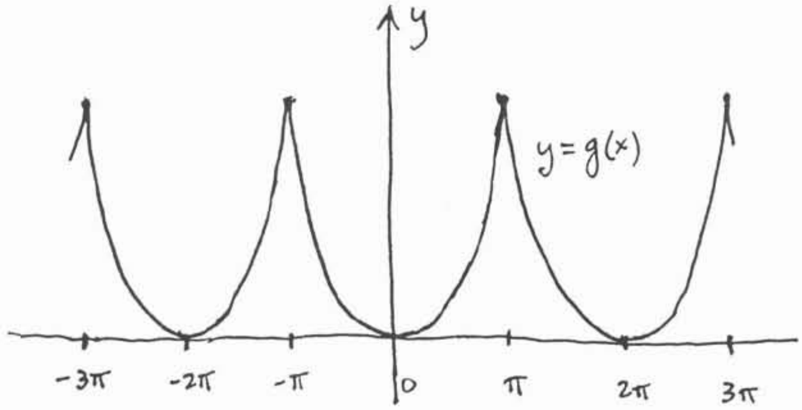
$$a_n = \frac{\langle f, \cos(n\cdot)\rangle}{\langle \cos(n\cdot), \cos(n\cdot)\rangle} = \frac{2}{\pi}\int_0^{\pi} \underbrace{x^2}_{u}\underbrace{\cos(nx)\,dx}_{dv} = \frac{2}{\pi}\left[\cancelto{0}{\left.\frac{x^2\sin(nx)}{n}\right|_0^{\pi}} - \frac{1}{n}\int_0^{\pi} \overbrace{2x}^{u}\,\overbrace{\sin(nx)\,dx}^{dv} \right]$$

$$= \frac{-2}{n\pi}\left[\left.\frac{-2x\cos(nx)}{n}\right|_0^{\pi} - \cancelto{0}{\int_0^{\pi} \frac{-\cos(nx)}{n}\,2\,dx} \right] = \frac{4\cos(n\pi)}{n^2} = \frac{4(-1)^n}{n^2} \quad \text{for } n \ge 1.$$

Therefore the Fourier cosine series of $f(x) = x^2$ on $[0,\pi]$ is

$$x^2 \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}.$$

(b) Extend f to $[-\pi,\pi]$ as an even function: $f_{ext}(x) = x^2$ on $[-\pi,\pi]$ and then extend f_{ext} to the whole real line as a 2π-periodic function g. The graph of g is shown below:



$g(x) = x^2$ if $-\pi \le x \le \pi$
and
$g(x) = g(x+2\pi)$ for all real x.

Clearly g is continuous on $(-\infty,\infty)$ and g' is piecewise continuous on $(-\infty,\infty)$. By Theorem 4^{∞}, the classical full Fourier series of g at x converges to $\frac{g(x^+)+g(x^-)}{2}$ for every real x, and this is equal to $g(x)$ by continuity. But

g is an even function so the classical full Fourier series of g is a Fourier cosine series. Because $g(x) = f(x)$ on $[0,\pi]$, the full Fourier series of g agrees with the Fourier cosine series of f on $[0,\pi]$.

$\boxed{\text{Hence the Fourier cosine series of } f \text{ at } x \text{ converges pointwise to } f(x) = x^2 \text{ on } [0,\pi].}$

6 pts. (c) By parts (b) and (a), $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}$ for all $0 \le x \le \pi$.

Therefore

$$\max_{0\le x\le\pi} \left| x^2 - \left(\frac{\pi^2}{3} + \sum_{n=1}^{N} \frac{4(-1)^n\cos(nx)}{n^2}\right)\right| = \max_{0\le x\le\pi} \left| \sum_{n=N+1}^{\infty} \frac{4(-1)^n\cos(nx)}{n^2}\right|$$

$$\le \max_{0\le x\le\pi} \sum_{n=N+1}^{\infty} \left| \frac{4(-1)^n\cos(nx)}{n^2}\right| \le 4\sum_{n=N+1}^{\infty} \frac{1}{n^2} \longrightarrow 0 \text{ as } N\to\infty$$

since $\sum_{n=N+1}^{\infty} \frac{1}{n^2}$ is a "tail" of the convergent p-series: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $(p=2)$.

Therefore the Fourier partial sums $S_N f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{N} \frac{4(-1)^n\cos(nx)}{n^2}$ $(N=0,1,2,\dots)$ converge uniformly to $f(x) = x^2$ on $[0,\pi]$.

7 pts. (d) Take $x=0$ in the identity $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n\cos(nx)}{n^2}$ for $0\le x\le\pi$, to get $0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ and hence $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \boxed{\frac{-\pi^2}{12}}$. We apply Parseval's identity to the Fourier cosine series: $|a_0|^2\int_0^{\pi} 1^2\,dx + \sum_{n=1}^{\infty}|a_n|^2\int_0^{\pi}\cos^2(nx)\,dx = \int_0^{\pi}|f(x)|^2\,dx$;

so $\left(\frac{\pi^2}{3}\right)^2\cdot\pi + \sum_{n=1}^{\infty}\left|\frac{4(-1)^n}{n^2}\right|^2\cdot\frac{\pi}{2} = \frac{\pi^5}{5} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{\pi^5}{5} - \frac{\pi^5}{9}\right)\frac{2}{16\pi} = \boxed{\frac{\pi^4}{90}}$

$\boxed{\#6}$ (a) We seek nontrivial solutions of the form $u(x,t) = X(x)T(t)$ to (1)-(2)-(3)-(4). Substituting in (1) yields

$$X(x)T''(t) - X''(x)T(t) + 2X(x)T'(t) = 0$$

or

$$-\frac{X''(x)}{X(x)} = -\frac{(T''(t) + 2T'(t))}{T(t)} = \text{constant} = \lambda .$$

Substituting in (2) and (3) gives $X'(0)T(t) = 0 = X'(\pi)T(t)$ for $t \geq 0$ while (4) implies $X(x)T(0) = 0$ for $0 \leq x \leq \pi$. Thus

3 pts to here

$$\begin{cases} X''(x) + \lambda X(x) = 0, \; X'(0) = 0 = X'(\pi), \\ T''(t) + 2T'(t) + \lambda T(t) = 0, \; T(0) = 0 . \end{cases}$$

6 pts to here

By Sec. 4.2, the eigenvalues and eigenfunctions in the presence of Neumann boundary conditions are $\lambda_n = n^2$ and $X_n(x) = \cos(nx)$ $(n = 0, 1, 2, \ldots)$. Substituting $\lambda = \lambda_n = n^2$ in the T-equations yields

$$T_n''(t) + 2T_n'(t) + n^2 T_n(t) = 0, \; T_n(0) = 0 .$$

We look for solutions to the differential equation of the form $T_n(t) = e^{r_n t}$. This leads to

$$r_n^2 + 2r_n + n^2 = 0 \quad \Rightarrow \quad r_n = -1 \pm \sqrt{1-n^2} .$$

If $n = 0$ then $r_0 = 0, -2$ so $T_0(t) = c_1 + c_2 e^{-2t}$.

If $n = 1$ then $r_1 = -1$ (multiplicity two) so $T_1(t) = c_1 e^{-t} + c_2 t e^{-t}$.

If $n \geq 2$ then $r_n = -1 \pm i\sqrt{n^2-1}$ so $T_n(t) = e^{-t}\left(c_1 \cos(\sqrt{n^2-1}\, t) + c_2 \sin(\sqrt{n^2-1}\, t)\right)$.

Taking into account $T_n(0) = 0$, we have, up to a constant factor,

9 pts to here

$$T_0(t) = 1 - e^{-2t}, \quad T_1(t) = te^{-t}, \quad T_n(t) = e^{-t}\sin(\sqrt{n^2-1}\, t) \quad (n \geq 2) .$$

By superposition, a formal solution to (1)-(2)-(3)-(4) is

$$u(x,t) = \sum_{n=0}^{\infty} A_n X_n(x) T_n(t) = A_0(1-e^{-2t}) + A_1\cos(x)te^{-t} + \sum_{n=2}^{\infty} A_n \cos(nx) e^{-t} \sin(\sqrt{n^2-1}\,t).$$

Then

$$u_t(x,t) = 2A_0 e^{-2t} + A_1\cos(x)(1-t)e^{-t} + \sum_{n=2}^{\infty} A_n\cos(nx)e^{-t}\left[\sqrt{n^2-1}\cos(\sqrt{n^2-1}\,t) - \sin(\sqrt{n^2-1}\,t)\right]$$

so to satisfy (5) we must have

$$x^2 = u_t(x,0) = 2A_0 + A_1\cos(x) + \sum_{n=2}^{\infty} A_n\sqrt{n^2-1}\cos(nx) \quad \text{for } 0 \le x \le \pi.$$

By problem #5, it follows that

$$2A_0 = \frac{\pi^2}{3}, \quad A_1 = -4, \quad \text{and} \quad A_n\sqrt{n^2-1} = \frac{4(-1)^n}{n^2} \quad \text{for } n \ge 2.$$

Thus, a solution to (1)-(2)-(3)-(4)-(5) is

15 pts to here.

$$\boxed{u(x,t) = \frac{\pi^2}{6}(1-e^{-2t}) - 4\cos(x)te^{-t} + 4e^{-t}\sum_{n=2}^{\infty} \frac{(-1)^n\cos(nx)\sin(\sqrt{n^2-1}\,t)}{n^2\sqrt{n^2-1}}}$$

5 pts

(b) Let $u = u(x,t)$ satisfy (1)-(2)-(3) and consider the energy function

$E(t) = \frac{1}{2}\int_0^{\pi}\left[u_t^2(x,t) + u_x^2(x,t)\right]dx$ of the solution u for $0 \le t < \infty$.

Then $E'(t) = \frac{1}{2}\int_0^{\pi} \frac{\partial}{\partial t}\left[u_t^2(x,t) + u_x^2(x,t)\right]dx$

$$= \int_0^{\pi}\left[u_t(x,t)u_{tt}(x,t) + \overbrace{u_x(x,t)}^{v}\,\overbrace{u_{xt}(x,t)}^{dV}\right]dx$$

$$= \int_0^{\pi} u_t(x,t)u_{tt}(x,t)\,dx + \cancel{u_x(x,t)u_t(x,t)\Big|_{x=0}^{\pi}}^{\,0 \text{ by } (2)-(3)} - \int_0^{\pi} u_t(x,t)u_{xx}(x,t)\,dx$$

$$= \int_0^{\pi} u_t(x,t)\left[u_{tt}(x,t) - u_{xx}(x,t)\right]dx$$

$$= -2\int_0^{\pi} u_t^2(x,t)\,dx \qquad (\text{by } (1))$$

$$\le 0.$$

Consequently $E = E(t)$ is decreasing on $t \geq 0$.

5 pts (c) Suppose that $u = v(x,t)$ is another solution to the problem in part (a) and consider $w(x,t) = u(x,t) - v(x,t)$ for $0 \leq x \leq \pi$ and $t \geq 0$. Then w solves

$$\text{(1')} \qquad w_{tt} - w_{xx} + 2w_t = 0 \quad \text{in } 0 < x < \pi,\ 0 < t < \infty,$$

$$\text{(2')-(3')} \qquad w_x(0,t) = 0 = w_x(\pi,t) \quad \text{for } t \geq 0,$$

$$\text{(4')-(5')} \qquad w(x,0) = 0 = w_t(x,0) \quad \text{for } 0 \leq x \leq \pi.$$

By part (b), the energy function

$$E(t) = \frac{1}{2}\int_0^\pi \left[w_t^2(x,t) + w_x^2(x,t) \right] dx$$

is decreasing on $0 \leq t < \infty$. But for all $t \geq 0$,

$$0 \leq E(t) \leq E(0) = \frac{1}{2}\int_0^\pi \left[w_t^2(x,0) + w_x^2(x,0) \right] dx = 0$$

by (5') and differentiation of (4'). Thus the vanishing theorem implies $w_t(x,t) = 0 = w_x(x,t)$ for $0 \leq x \leq \pi$ and $0 \leq t < \infty$ so $w(x,t) =$ constant in the strip. But $w(x,0) = 0$ for all $0 \leq x \leq \pi$ ((4')) so $w(x,t) = 0$ in the strip. That is, $v(x,t) = u(x,t)$ for all $0 \leq x \leq \pi$ and $0 \leq t < \infty$. This shows that there is only one solution to the problem in (a).

$\boxed{\#7}$ (a) If $u = u(r;\theta)$ is the steady-state temperature of the material at the point $(r;\theta)$ in the disk $r \le 1$, then u solves

$$\begin{cases} \nabla^2 u = 0 & \text{if } r < 1, \\ u(1;\theta) = \begin{cases} 50 & \text{if } 0 < \theta < \pi, \\ -50 & \text{if } -\pi < \theta < 0. \end{cases} \end{cases}$$

(b) By Poisson's formula, for $0 \le r < 1$ and $-\pi \le \theta \le \pi$ we have

$$u(r;\theta) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{h(\varphi)\,d\varphi}{1-2r\cos(\theta-\varphi)+r^2} = \frac{1-r^2}{2\pi}\left(\int_0^{\pi} \frac{50\,d\varphi}{1-2r\cos(\varphi-\theta)+r^2} + \int_{-\pi}^{0} \frac{-50\,d\varphi}{1-2r\cos(\varphi-\theta)+r^2}\right)$$

$$= \frac{50(1-r^2)}{2\pi}\left[\int_{-\theta}^{\pi-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)} - \int_{-\pi-\theta}^{-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)}\right].$$

(c) By the mean value property of harmonic functions,

5 pts. $$u(0;\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(1;\varphi)\,d\varphi = \frac{1}{2\pi}\left(\int_0^{\pi} 50\,d\varphi + \int_{-\pi}^{0} -50\,d\varphi\right) = 0.$$

10 pts. → (d) We use the integration formula

$$\int_0^x \frac{d\psi}{a+b\cos(\psi)} = \frac{2}{\sqrt{a^2-b^2}}\operatorname{Arctan}\left(\sqrt{\frac{a-b}{a+b}}\tan(x/2)\right) + \frac{2\pi}{\sqrt{a^2-b^2}}\cdot \operatorname{floor}\left(\frac{x}{2\pi}+\frac{1}{2}\right)$$

which is valid if $a > |b|$ and which is available from a standard graphical calculator (e.g. HP-49G). In particular,

$$(*)\quad \int_0^x \frac{d\psi}{a+b\cos(\psi)} = \begin{cases} \frac{2}{\sqrt{a^2-b^2}}\operatorname{Arctan}\left(\sqrt{\frac{a-b}{a+b}}\tan(x/2)\right) + \frac{2\pi}{\sqrt{a^2-b^2}} & \text{if } \pi < x < 3\pi, \\ \frac{2}{\sqrt{a^2-b^2}}\operatorname{Arctan}\left(\sqrt{\frac{a-b}{a+b}}\tan(x/2)\right) & \text{if } -\pi < x < \pi, \\ \frac{2}{\sqrt{a^2-b^2}}\operatorname{Arctan}\left(\sqrt{\frac{a-b}{a+b}}\tan(x/2)\right) - \frac{2\pi}{\sqrt{a^2-b^2}} & \text{if } -3\pi < x < -\pi. \end{cases}$$

In (*), set $a = 1+r^2$ and $b = -2r$. Since $0 \le r < 1$, we have

$$\sqrt{a^2-b^2} = \sqrt{(1+r^2)^2 - 4r^2} = \sqrt{1-2r^2+r^4} = 1-r^2$$

and

$$\sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{1+r^2+2r}{1+r^2-2r}} = \frac{1+r}{1-r},$$

so (*) becomes

$$(**) \quad \int_0^x \frac{d\psi}{1+r^2-2r\cos(\psi)} = \begin{cases} \frac{2}{1-r^2}\operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(x/2)\right) + \frac{2\pi}{1-r^2} & \text{if } \pi < x < 3\pi, \\ \frac{2}{1-r^2}\operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(x/2)\right) & \text{if } -\pi < x < \pi, \\ \frac{2}{1-r^2}\operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(x/2)\right) - \frac{2\pi}{1-r^2} & \text{if } -3\pi < x < -\pi. \end{cases}$$

Suppose $0 < \theta < \pi$. Then $-\pi < -\theta < \pi - \theta < \pi$ so (**) implies

$$(\dagger) \quad \int_{-\theta}^{\pi-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)} = \int_0^{\pi-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)} - \int_0^{-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)}$$

$$= \frac{2}{1-r^2}\left[\operatorname{Arctan}\left(\frac{1+r}{1-r}\tan\left(\frac{\pi-\theta}{2}\right)\right) - \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(-\theta/2)\right)\right]$$

$$= \frac{2}{1-r^2}\left[\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right)\right].$$

Also $-2\pi < -\pi-\theta < -\pi < -\theta < 0$ so from (**) it follows that

$$(\dagger\dagger) \quad \int_{-\pi-\theta}^{-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)} = -\int_0^{-\pi-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)} + \int_0^{-\theta} \frac{d\psi}{1+r^2-2r\cos(\psi)}$$

$$= \frac{2}{1-r^2}\left[-\operatorname{Arctan}\left(\frac{1+r}{1-r}\tan\left(\frac{-\pi-\theta}{2}\right)\right) + \pi + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan\left(-\frac{\theta}{2}\right)\right)\right]$$

$$= \frac{-2}{1-r^2}\left[\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) - \pi + \operatorname{Artan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right)\right].$$

Using (b), it follows from (†) and (††) that

$$u(r;\theta) = \frac{100}{\pi}\left\{\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right) - \frac{\pi}{2}\right\}$$

for $0<\theta<\pi$. A simple change of variables in the formula of part (b) shows that $u(r;-\theta) = u(r;\theta)$, so for $-\pi<\theta<0$ we have

$$\begin{aligned} u(r;\theta) &= -u(r;-\theta) \\ &= -\frac{100}{\pi}\left\{\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(-\theta/2)\right) + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(-\theta/2)\right) - \frac{\pi}{2}\right\} \\ &= \frac{100}{\pi}\left\{\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right) + \frac{\pi}{2}\right\}. \end{aligned}$$

In summary,

$$\boxed{u(r;\theta) = \begin{cases} \frac{100}{\pi}\left[\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right) - \frac{\pi}{2}\right] & \text{if } 0<\theta<\pi \\ \frac{100}{\pi}\left[\operatorname{Arctan}\left(\frac{1+r}{1-r}\cot(\theta/2)\right) + \operatorname{Arctan}\left(\frac{1+r}{1-r}\tan(\theta/2)\right) + \frac{\pi}{2}\right] & \text{if } -\pi<\theta< \end{cases}}$$

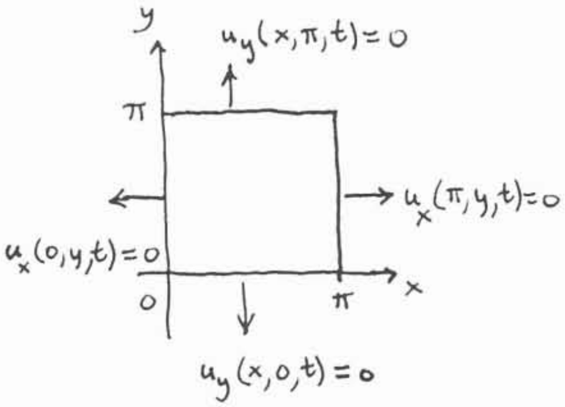
or equivalently

$$\boxed{u(r;\theta) = \frac{100}{\pi}\operatorname{Arctan}\left(\frac{2r\sin(\theta)}{1-r^2}\right)}.$$

Hints: $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta}{1-\tan(\alpha)\tan(}$

and $\tan(\alpha)+\cot(\alpha) = \frac{2}{\sin(2\alpha)}$

$\boxed{\#8}$ (a) The temperature function $u = u(x,y,t)$ solves $u_t - \nabla^2 u = 0$ in the Square $S: 0<x<\pi,\ 0<y<\pi$ for all $t>0$ and satisfies the boundary condition $\frac{\partial u}{\partial n} = 0$ on ∂S for all $t>0$ and the initial condition $u(x,y,0) = \cos^2(x)\cos^2(y)$ on S. This can be described equivalently as follows:



$$\begin{cases} u_t - (u_{xx}+u_{yy}) \overset{①}{=} 0 & \text{if } 0<x<\pi,\ 0<y<\pi,\ 0<t<\infty \\ u_x(0,y,t) \overset{②}{=} 0 \overset{③}{=} u_x(\pi,y,t) & \text{if } 0\le y\le \pi,\ 0\le t<\infty, \\ u_y(x,0,t) \overset{④}{=} 0 \overset{⑤}{=} u_y(x,\pi,t) & \text{if } 0\le x\le\pi,\ 0\le t<\infty, \\ u(x,y,0) \overset{⑥}{=} \cos^2(x)\cos^2(y) & \text{if } 0\le x\le\pi,\ 0\le y\le\pi. \end{cases}$$

We seek nontrivial solutions of ①-②-③-④-⑤ of the form $u(x,y,t) = X(x)Y(y)T(t)$. Substituting in ① yields

$$X(x)Y(y)T'(t) - \Big(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)\Big) = 0.$$

Dividing through by $X(x)Y(y)T(t)$ and rearranging gives

$$(*) \qquad \frac{-X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} - \frac{T'(t)}{T(t)} = \text{constant} = \lambda$$

and then

$$(**) \qquad \frac{-Y''(y)}{Y(y)} = \frac{-T'(t)}{T(t)} - \lambda = \text{constant} = \mu.$$

Substituting in ②-③ and ④-⑤ yields

$$(***) \qquad \begin{cases} X'(0)Y(y)T(t) = 0 = X'(\pi)Y(y)T(t) & \text{for all } 0\le y\le\pi,\ 0\le t<\infty, \\ X(x)Y'(0)T(t) = 0 = X(x)Y'(\pi)T(t) & \text{for all } 0\le x\le\pi,\ 0\le t<\infty. \end{cases}$$

Combining (*), (**), and (***) leads to

$$⑦ \quad X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(\pi)$$

5 pts

$$⑧ \quad Y''(y) + \mu Y(y) = 0, \quad Y'(0) = 0 = Y'(\pi)$$

$$⑨ \quad T'(t) + (\lambda+\mu)T(t) = 0$$

From Sec. 4.2, the eigenvalues and eigenfunctions for ⑦ and ⑧ are

2 pts.

$$\lambda_\ell = \ell^2 \text{ and } X_\ell(x) = \cos(\ell x) \quad (\ell = 0,1,2,\dots)$$

2 pts.

$$\mu_m = m^2 \text{ and } Y_m(y) = \cos(my) \quad (m = 0,1,2,\dots).$$

Substitituting $\lambda_\ell = \ell^2$ and $\mu_m = m^2$ into ⑨ and solving yields

1 pt.

$$T_{\ell,m}(t) = e^{-(\ell^2+m^2)t},$$

up to a constant factor. Superposition implies

$$u(x,y,t) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m} X_\ell(x) Y_m(y) T_{\ell,m}(t)$$

2 pts.

$$= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m} \cos(\ell x)\cos(my) e^{-(\ell^2+m^2)t}$$

is a formal solution to ①-②-③-④-⑤ for arbitrary constants $A_{\ell,m}$.
To satisfy ⑥ we need

$$\left[\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right]\left[\tfrac{1}{2} + \tfrac{1}{2}\cos(2y)\right] = \cos^2(x)\cos^2(y) = u(x,y,0) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m}\cos(\ell x)\cos(my)$$

for all $0 \le x \le \pi$, $0 \le y \le \pi$. Expanding the left member above,

$$\frac{1}{4} + \frac{1}{4}\cos(2x) + \frac{1}{4}\cos(2y) + \frac{1}{4}\cos(2x)\cos(2y) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} A_{l,m}\cos(lx)\cos(my)$$

for all $0 \le x \le \pi$, $0 \le y \le \pi$. By inspection we have a solution if

$$A_{0,0} = A_{2,0} = A_{0,2} = A_{2,2} = \frac{1}{4}$$

and all other $A_{l,m} = 0$. Therefore

3 pts.

$$\boxed{u(x,y,t) = \frac{1}{4} + \frac{1}{4}\cos(2x)e^{-4t} + \frac{1}{4}\cos(2y)e^{-4t} + \frac{1}{4}\cos(2x)\cos(2y)e^{-8t}}$$

(3 pts to here.)

solves ①-②-③-④-⑤-⑥.

8 pts. (b) Let $u = v(x,y,t)$ be another solution to ①-②-③-④-⑤-⑥. Then $w(x,y,t) = u(x,y,t) - v(x,y,t)$ solves

$$\begin{cases} w_t - (w_{xx} + w_{yy}) \overset{1'}{=} 0 & \text{if } 0<x<\pi,\ 0<y<\pi,\ 0<t<\infty, \\ w_x(0,y,t) \overset{2'}{=} 0 \overset{3'}{=} w_x(\pi,y,t) & \text{if } 0 \le y \le \pi,\ 0 \le t < \infty, \\ w_y(x,0,t) \overset{4'}{=} 0 \overset{5'}{=} w_y(x,\pi,t) & \text{if } 0 \le x \le \pi,\ 0 \le t < \infty, \\ w(x,y,0) \overset{6'}{=} 0 & \text{if } 0 \le x \le \pi,\ 0 \le y \le \pi. \end{cases}$$

Consider the mean-square temperature at t of w:

$$\Theta(t) = \left(\int_0^{\pi}\int_0^{\pi} w^2(x,y,t)\,dx\,dy\right)^{1/2}.$$

Squaring and differentiating yields

$$2\,\Theta(t)\,\Theta'(t) = \int_0^{\pi}\int_0^{\pi} \frac{\partial}{\partial t}\left(w^2(x,y,t)\right)dx\,dy = 2\int_0^{\pi}\int_0^{\pi} w(x,y,t)\,w_t(x,y,t)\,dx\,dy$$

and hence by (1'),

$$\Theta(t)\,\Theta'(t) = \int_0^{\pi}\int_0^{\pi} w(x,y,t)\left[w_{xx}(x,y,t) + w_{yy}(x,y,t)\right]dx\,dy$$

$$= \int_0^{\pi}\left(\int_0^{\pi} \overbrace{w(x,y,t)}^{u}\,\overbrace{w_{xx}(x,y,t)\,dx}^{dv}\right)dy + \int_0^{\pi}\left(\int_0^{\pi} \overbrace{w(x,y,t)}^{u}\,\overbrace{w_{yy}(x,y,t)\,dy}^{dv}\right)dx$$

$$= \int_0^{\pi}\left(w(x,y,t)\,w_x(x,y,t)\Big|_{x=0}^{\pi} - \int_0^{\pi} w_x^2(x,y,t)\,dx\right)dy$$

$$+ \int_0^{\pi}\left(w(x,y,t)\,w_y(x,y,t)\Big|_{y=0}^{\pi} - \int_0^{\pi} w_y^2(x,y,t)\,dy\right)dx .$$

Applying (2')-(3') and (4')-(5') we see that

$$\Theta(t)\,\Theta'(t) = -\int_0^{\pi}\int_0^{\pi}\left[w_x^2(x,y,t) + w_y^2(x,y,t)\right]dx\,dy \le 0 .$$

Since $\Theta(t) \ge 0$, it follows that $\Theta'(t) \le 0$ for all $t \ge 0$, and hence Θ is a decreasing function on $[0,\infty)$. Therefore

$$0 \le \Theta(t) \le \Theta(0) = \left(\int_0^{\pi}\int_0^{\pi} w^2(x,y,0)\,dx\,dy\right)^{1/2} = 0$$

by (6'). Consequently $\Theta(t) = \left(\int_0^{\pi}\int_0^{\pi} w^2(x,y,t)\,dx\,dy\right)^{1/2} = 0$ for all $t \ge 0$, so by the vanishing theorem

$$w(x,y,t) = 0 \quad \text{for all } 0 \le x \le \pi,\ 0 \le y \le \pi,\ 0 \le t < \infty .$$

That is, $v(x,y,t) = u(x,y,t)$ for all $0 \le x \le \pi$, $0 \le y \le \pi$, $0 \le t < \infty$

and $\boxed{\text{the solution to the problem in (a) is unique.}}$

5 pts. (c) The steady-state temperature distribution in the square is constant:

$$U(x,y) = \lim_{t\to\infty} u(x,y,t)$$

$$= \lim_{t\to\infty} \left(\frac{1}{4} + \frac{1}{4}\cos(2x)e^{-4t} + \frac{1}{4}\cos(2y)e^{-4t} + \frac{1}{4}\cos(2x)\cos(2y)e^{-8t} \right),$$

$$= \boxed{\frac{1}{4}}.$$

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$
A.	$\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\dfrac{2}{\pi}}\dfrac{\sin(b\xi)}{\xi}$
B.	$\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\dfrac{1}{x^2+a^2}\quad (a>0)$	$\sqrt{\dfrac{\pi}{2}}\dfrac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\dfrac{2}{\pi}}\dfrac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\dfrac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2}\quad (a>0)$	$\dfrac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\dfrac{\sin(ax)}{x}\quad (a>0)$	$\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\dfrac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$

Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b \text{ with any symmetric boundary conditions}$$

and let $\Phi=\{X_1,X_2,X_3,...\}$ be the complete orthogonal set of eigenfunctions for (1). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$f(x)\sim\sum_{n=1}^{\infty}A_nX_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle}\quad (n=1,2,3,...).$$

Theorem 2. (Uniform Convergence) If
(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and
(ii) f satisfies the given symmetric boundary conditions,
then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)
(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.
(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty}A_nX_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

Math 325

Final Exam

Summer 2010

Number taking exam: 20

Mean: 116.6

Standard Deviation: 33.4

Median: 117.5

Distribution of Scores:

	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
174 - 200	A	A	2
146 - 173	B	B	1
120 - 145	C	B	5
100 - 119	C	C	4
0 - 99	F	D	8