2014 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Administered by the Mathematical Association of America

Problem A1

Prove that every nonzero coefficient of the Taylor series of

$$\left(1-x+x^2\right)e^x$$

about $x=0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Problem A2

Let A be the $n\times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i,j)}$$

for $1\le i,j\le n$. Compute $\det(A)$.

Problem A3

Let $a_0=5/2$ and $a_k=a_{k-1}^2-2$ for $k\ge 1$. Compute

$$\prod_{k=0}^{\infty}\left(1-\frac{1}{a_k}\right)$$

in closed form.

Problem A4

Suppose X is a random variable that takes on only nonnegative integer values, with $E[X]=1$, $E\left[X^2\right]=2$, and $E\left[X^3\right]=5$. (Here $E[y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X=0$.

Problem A5

Let

$$P_n(x)=1+2x+3x^2+\cdots+nx^{n-1}.$$

Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j\ne k$.

Problem A6

Let n be a positive integer. What is the largest k for which there exist $n\times n$ matrices $M_1,\ldots,M_k$ and $N_1,\ldots,N_k$ with real entries such that for all i and j, the matrix product M_iN_j has a zero entry somewhere on its diagonal if and only if $i\ne j$?

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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Problem B1

A *base* 10 *over-expansion* of a positive integer N is an expression of the form
$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$
with $d_k \neq 0$ and $d_i \in \{0,1,2,\ldots,10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

Problem B2

Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

Problem B3

Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m+n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

Problem B4

Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

Problem B5

In the 75th Annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(1) A player cannot choose an element that has been chosen by either player on any previous turn.

(2) A player can only choose an element that commutes with all previously chosen elements.

(3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy?
(Your answer may depend on n and p.)

Problem B6

Let $f : [0,1] \to \mathbb{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in [0,1]$. Suppose also that for each rational number $r \in [0,1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \ldots, I_n$ such that f is a linear function on each I_i and $[0,1] = \bigcup_{i=1}^{n} I_i$.

SEVENTY FOURTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 7, 2013

Examination A

Problem A1

Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Problem A2

Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

Problem A3

Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Problem A4

A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \ge 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that $|Z(w) - Z(w')| \le 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) \text{ and } N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

are both integers. Prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.

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Saturday, December 7, 2013

Problem A5

For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}\left(\Delta A_i A_j A_k\right) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

Problem A6

Define a function $w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a,b)$ be as in the table shown; otherwise, let $w(a,b) = 0$.

$w(a,b)$		b = −2	−1	0	1	2
	−2	−1	−2	2	−2	−1
	−1	−2	4	−4	4	−2
a	0	2	−4	12	−4	2
	1	−2	4	−4	4	−2
	2	−1	−2	2	−2	−1

For every finite subset S of $\mathbb{Z} \times \mathbb{Z}$, define

$$A(S) = \sum_{(\mathbf{s},\mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}').$$

Prove that if S is any finite nonempty subset of $\mathbb{Z} \times \mathbb{Z}$, then $A(S) > 0$. (For example, if $S = \{(0,1),(0,2),(2,0),(3,1)\}$, then the terms in $A(S)$ are 12, 12, 12, 12, 4, 4, 0, 0, 0, 0, $-1, -1, -2, -2, -4, -4$.)

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Saturday, December 7, 2013 — Examination B

Problem B1

For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1)=1$, $c(2n)=c(n)$, and $c(2n+1)=(-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Problem B2

Let $C=\bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x)=1+\sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(*i*) $f(x)\geq 0$ for all real x, and

(*ii*) $a_n=0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

Problem B3

Let P be a nonempty collection of subsets of $\{1,\ldots,n\}$ such that:

(*i*) if $S,S'\in P$, then $S\cup S'\in P$ and $S\cap S'\in P$, and

(*ii*) if $S\in P$ and $S\neq\varnothing$, then there is a subset $T\subset S$ such that $T\in P$ and T contains exactly one fewer element than S.

Suppose that $f:P\to\mathbb{R}$ is a function such that $f(\varnothing)=0$ and

$$f(S\cup S')=f(S)+f(S')-f(S\cap S') \text{ for all } S,S'\in P.$$

Must there exist real numbers $f_1,\ldots,f_n$ such that

$$f(S)=\sum_{i\in S} f_i$$

for every $S\in P$?

SEVENTY FOURTH ANNUAL
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Saturday, December 7, 2013 Examination B

Problem B4

For any continuous real-valued function f defined on the interval $[0,1]$, let

$$\mu(f)=\int_0^1 f(x)\,dx,\quad \operatorname{Var}(f)=\int_0^1 \left(f(x)-\mu(f)\right)^2 dx,\quad M(f)=\max_{0\le x\le 1}|f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0,1]$, then

$$\operatorname{Var}(fg)\le 2\operatorname{Var}(f)M(g)^2+2\operatorname{Var}(g)M(f)^2.$$

Problem B5

Let $X=\{1,2,\ldots,n\}$, and let $k\in X$. Show that there are exactly $k\cdot n^{n-1}$ functions $f:X\to X$ such that for every $x\in X$ there is a $j\ge 0$ such that $f^{(j)}(x)\le k$.
$\left[\text{Here } f^{(j)} \text{ denotes the } j^{\text{th}} \text{ iterate of } f, \text{ so that } f^{(0)}(x)=x \text{ and } f^{(j+1)}(x)=f\left(f^{(j)}(x)\right).\right]$

Problem B6

Let $n\ge 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

SEVENTY-THIRD ANNUAL
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Saturday, December 1, 2012 — Examination A

Administered by the Mathematical Association of America

Problem A1

Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Problem A2

Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

Problem A3

Let $f : [-1,1] \to \mathbb{R}$ be a continuous function such that

(i) $f(x) = \dfrac{2-x^2}{2} f\left(\dfrac{x^2}{2-x^2}\right)$ for every x in $[-1,1]$,

(ii) $f(0) = 1$, and

(iii) $\lim\limits_{x \to 1^-} \dfrac{f(x)}{\sqrt{1-x}}$ exists and is finite.

Prove that f is unique, and express $f(x)$ in closed form.

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Saturday, December 1, 2012
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Problem A4

Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b + mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is the intersection of A with some arithmetic progression.

Problem A5

Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries in $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \ldots, p^n$ are distinct.

Problem A6

Let $f(x, y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x, y)$ over R equals 0. Must $f(x, y)$ be identically 0?

SEVENTY-THIRD ANNUAL
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Saturday, December 1, 2012

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Examination B

Problem B1

Let S be a class of functions from $[0,\infty)$ to $[0,\infty)$ that satisfies:

(i) The functions $f_1(x)=e^x-1$ and $f_2(x)=\ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x)+g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x)\geq g(x)$ for all $x\geq 0$, then the function $f(x)-g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

Problem B2

Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P)>0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n>c(P)/V^2$.

Problem B3

A round-robin tournament among $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

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Saturday, December 1, 2012
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Examination B

Problem B4

Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0,1,2,\ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

Problem B5

Prove that, for any two bounded functions $g_1, g_2 : \mathbb{R} \to [1,\infty)$, there exist functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that, for every $x \in \mathbb{R}$,

$$\sup_{s\in\mathbb{R}}\left(g_1(s)^x g_2(s)\right) = \max_{t\in\mathbb{R}}\left(xh_1(t) + h_2(t)\right).$$

Problem B6

Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

SEVENTY SECOND ANNUAL

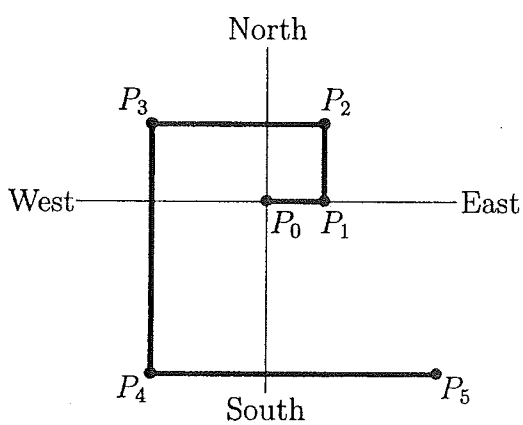
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 3, 2011 — Examination A

Problem A1

Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0,0), P_1, \dots, P_n$ such that $n \ge 2$ and:

- The directed line segments $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.
- The lengths of these line segments are positive and strictly increasing.



How many of the points (x, y) with integer coordinates $0 \le x \le 2011$, $0 \le y \le 2011$ *cannot* be the last point, P_n of any growing spiral?

Problem A2

Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $\mathrm{b}_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S.

Problem A3

Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x \, dx}{\int_0^{\pi/2} x^r \cos x \, dx} = L.$$

SEVENTY SECOND ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 3, 2011

Problem A4

For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

Problem A5

Let $F:\mathbb{R}^2 \to \mathbb{R}$ and $g:\mathbb{R} \to \mathbb{R}$ be twice continuously differentiable functions with the following properties:

- $F(u,u)=0$ for every $u \in \mathbb{R}$;
- for every $x \in \mathbb{R}$, $g(x)>0$ and $x^2 g(x) \le 1$;
- for every $(u,v) \in \mathbb{R}^2$, the vector $\nabla F(u,v)$ is either $\mathbf{0}$ or parallel to the vector $\langle g(u), -g(v)\rangle$.

Prove that there exists a constant C such that for every $n \ge 2$ and any $x_1,\ldots,x_{n+1} \in \mathbb{R}$, we have

$$\min_{i \neq j} \left|F(x_i,x_j)\right| \le \frac{C}{n}.$$

Problem A6

Let G be an abelian group with n elements, and let

$$\{g_1=e, g_2,\ldots,g_k\} \subsetneq G$$

be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, \ldots, g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$.

Prove that there exists a real number $b \in (0,1)$ such that

$$\lim_{m\to\infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\text{Prob}(g=x) - \frac{1}{n}\right)^2$$

is positive and finite.

SEVENTY SECOND ANNUAL
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Saturday, December 3, 2011 Examination B

Problem B1

Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < \left|h\sqrt{m} - k\sqrt{n}\right| < 2\varepsilon.$$

Problem B2

Let S be the set of all ordered triples (p,q,r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

Problem B3

Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

Problem B4

In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = \left(T_{hk}\right)$ and $W = \left(W_{hk}\right)$. Initially, $T = W = 0$. After every game, for every (h,k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det\left(T + iW\right)$ is a non-negative integer divisible by 2^{2010}.

SEVENTY SECOND ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 3, 2011 Examination B

Problem B5

Let $a_1, a_2, \ldots$ be real numbers. Suppose that there is a constant A such that for all n,

$$\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n}\frac{1}{1+(x-a_i)^2}\right)^2 dx \le An.$$

Prove that there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n}\left(1+(a_i-a_j)^2\right) \ge Bn^3.$$

Problem B6

Let p be an odd prime. Show that for at least $(p+1)/2$ values of n in $\{0,1,2,\ldots,p-1\}$,

$$\sum_{k=0}^{p-1} k!n^k \quad \text{is not divisible by } p.$$

SEVENTY-FIRST ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 4, 2010

Examination
A

Problem A1

Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same?
[When $n = 8$, the example $\{1, 2, 3, 6\}$, $\{4, 8\}$, $\{5, 7\}$ shows that the largest k is *at least* 3.]

Problem A2

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Problem A3

Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \le M$ for all (x,y) in $\mathbb{R}^2$, then h is identically zero.

Problem A4

Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

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Saturday, December 4, 2010

Problem A5

Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for all $\mathbf{a}, \mathbf{b} \in G$.

Problem A6

Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x \to \infty} f(x) = 0$.

Prove that $\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx$ diverges.

SEVENTY-FIRST ANNUAL
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Saturday, December 4, 2010

Examination B

Problem B1

Is there an infinite sequence of real numbers $a_1, a_2, a_3, \dots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Problem B2

Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Problem B3

There are 2010 boxes labeled $B_1, B_2, \dots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Problem B4

Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

Problem B5

Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

SEVENTY-FIRST ANNUAL
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Saturday, December 4, 2010

Examination B

Problem B6

Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the k^{th} power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

SEVENTIETH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 5, 2009

Examination A

Problem A1

Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane.

Problem A2

Functions $f,\ g,\ h$ are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh},\quad f(0)=1,$$
$$g' = fg^2h + \frac{4}{fh},\quad g(0)=1,$$
$$h' = 3fgh^2 + \frac{1}{fg},\quad h(0)=1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

Problem A3

Let d_n be the determinant of the $n\times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1,\ \cos 2,\ \ldots,\ \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$ The argument of $\cos$ is always in radians, not degrees.)

Evaluate $\lim\limits_{n\to\infty} d_n$.

SEVENTIETH ANNUAL
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Saturday, December 5, 2009

Problem A4

Let S be a set of rational numbers such that

(a) $0 \in S$;

(b) If $x \in S$ then $x+1 \in S$ and $x-1 \in S$; and

(c) If $x \in S$ and $x \notin \{0,1\}$, then $\frac{1}{x(x-1)} \in S$.

Must S contain all rational numbers?

Problem A5

Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

Problem A6

Let $f:[0,1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y)\,dy$, $b = \int_0^1 f(1,y)\,dy$, $c = \int_0^1 f(x,0)\,dx$, and $d = \int_0^1 f(x,1)\,dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0) = b-a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0,y_0) = d-c.$$

SEVENTIETH ANNUAL
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Saturday, December 5, 2009

Examination B

Problem B1

Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example, $\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$.

Problem B2

A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

Problem B3

Call a subset S of $\{1,2,\ldots,n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1,2,\ldots,n\}$. [For instance, every subset of $\{1,2,3\}$ except $\{1,3\}$ is mediocre, so $A(3) = 7$.] Find all positive integers n such that $A(n+2) - 2A(n+1) + A(n) = 1$.

Problem B4

Say that a polynomial with real coefficents in two variables, x, y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

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Saturday, December 5, 2009 Examination B

Problem B5

Let $f:(1,\infty)\to\mathbb{R}$ be a differentiable function such that

$$f'(x)=\frac{x^2-\left(f(x)\right)^2}{x^2\left(\left(f(x)\right)^2+1\right)} \qquad \text{for all } x>1.$$

Prove that $\lim_{x\to\infty} f(x)=\infty$.

Problem B6

Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0=0$ and $a_{2009}=n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form $b \bmod c$ for some earlier positive terms b and c. [Here $b \bmod c$ denotes the remainder when b is divided by c, so $0 \le (b \bmod c) < c$.]

SIXTY-NINTH ANNUAL
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Saturday, December 6, 2008 — Examination A

Problem A1

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y)+f(y,z)+f(z,x)=0$ for all real numbers x, y, and z. Prove that there exists a function $g:\mathbb{R}\to\mathbb{R}$ such that $f(x,y)=g(x)-g(y)$ for all real numbers x and y.

Problem A2

Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Problem A3

Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j<k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j,a_k)$ and $\operatorname{lcm}(a_j,a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

Problem A4

Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases} x & \text{if } x\le e \\ xf(\ln x) & \text{if } x>e. \end{cases}$$

Does $\displaystyle\sum_{n=1}^{\infty}\frac{1}{f(n)}$ converge?

SIXTY-NINTH ANNUAL
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Saturday, December 6, 2008

Problem A5

Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

Problem A6

Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \ln|G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, 4, 4, 2 is a subsequence of 2, 4, 6, 4, 2, but 2, 2, 4 is not.)

SIXTY-NINTH ANNUAL
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Saturday, December 6, 2008

Examination B

Problem B1

What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

Problem B2

Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}.$$

Problem B3

What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

Problem B4

Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.

Problem B5

Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a,b) = 1$.) (Note: gcd means greatest common divisor.)

SIXTY-NINTH ANNUAL
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Saturday, December 6, 2008 — Examination B

Problem B6

Let n and k be positive integers. Say that a permutation σ of $\{1,2,\ldots,n\}$ is k-*limited* if $|\sigma(i)-i|\leq k$ for all i. Prove that the number of k-limited permutations of $\{1,2,\ldots,n\}$ is odd if and only if $n\equiv 0$ or $1 \pmod{2k+1}$.

SIXTY-EIGHTH ANNUAL
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Saturday, December 1, 2007

Examination A

Problem A1

Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

Problem A2

Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set S in the plane is called *convex* if for any two points in S the line segment connecting them is contained in S.)

Problem A3

Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Problem A4

A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

Problem A5

Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n+1$.

SIXTY-EIGHTH ANNUAL
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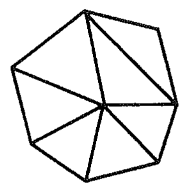
Saturday, December 1, 2007

Problem A6

A *triangulation* $\mathcal{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side of P is a side of exactly one triangle in $\mathcal{T}$. Say that $\mathcal{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. For example



Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

SIXTY-EIGHTH ANNUAL
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Saturday, December 1, 2007 — Examination B

Problem B1

Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n=1$.

Problem B2

Suppose that $f:[0,1]\to\mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx=0$. Prove that for every $\alpha\in(0,1)$,

$$\left|\int_0^\alpha f(x)\,dx\right|\le\frac{1}{8}\max_{0\le x\le 1}|f'(x)|.$$

Problem B3

Let $x_0=1$ and for $n\ge 0$, let $x_{n+1}=3x_n+\lfloor x_n\sqrt{5}\rfloor$. In particular, $x_1=5$, $x_2=26$, $x_3=136$, $x_4=712$. Find a closed-form expression for x_{2007}. ($\lfloor a\rfloor$ means the largest integer $\le a$.)

Problem B4

Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2+(Q(X))^2=X^{2n}+1$$

and $\deg P>\deg Q$.

Problem B5

Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \dots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor\frac{n}{k}\right\rfloor^k=P_0(n)+P_1(n)\left\lfloor\frac{n}{k}\right\rfloor+\cdots+P_{k-1}(n)\left\lfloor\frac{n}{k}\right\rfloor^{k-1}.$$

($\lfloor a\rfloor$ means the largest integer $\le a$.)

SIXTY-EIGHTH ANNUAL
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Saturday, December 1, 2007 — Examination B

Problem B6

For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1 \le k \le n$. Prove that for some constant C, independent of n,

$$n^{n^2/2 - Cn} e^{-n^2/4} \le f(n) \le n^{n^2/2 + Cn} e^{-n^2/4}.$$

SIXTY-SEVENTH ANNUAL

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 2, 2006

Examination A

Problem A1

Find the volume of the region of points (x, y, z) such that

$$\left(x^2 + y^2 + z^2 + 8\right)^2 \leq 36\left(x^2 + y^2\right).$$

Problem A2

Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n =$ 17, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

Problem A3

Let 1, 2, 3, …, 2005, 2006, 2007, 2009, 2012, 2016, … be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

Problem A4

Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i)	$\pi(k) > \pi(k+1)$	for $k = 1$
(ii)	$\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$	for $1 < k < n$
(iii)	$\pi(k-1) < \pi(k)$	for $k = n$

(For example, if $n = 5$ and π takes values at 1, 2, 3, 4, 5 of 2, 1, 4, 5, 3, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Problem A5

Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

Problem A6

Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

SIXTY-SEVENTH ANNUAL
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Saturday, December 2, 2006 — **Examination B**

Problem B1

Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are the vertices of an equilateral triangle, and find its area.

Problem B2

Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1}.$$

Problem B3

Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \varnothing$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

Problem B4

Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \le k \le n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.

SIXTY-SEVENTH ANNUAL
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Saturday, December 2, 2006 — Examination B

Problem B5

For each continuous function $f:[0,1]\to\mathbb{R}$, let $I(f)=\int_0^1 x^2 f(x)\,dx$ and $J(f)=\int_0^1 x\left(f(x)\right)^2 dx$. Find the maximum value of $I(f)-J(f)$ over all such functions f.

Problem B6

Let k be an integer greater than 1. Suppose $a_0>0$, and define

$$a_{n+1}=a_n+\frac{1}{\sqrt[k]{a_n}}$$

for $n\ge 0$. Evaluate

$$\lim_{n\to\infty}\frac{a_n^{k+1}}{n^k}.$$

SIXTY-SIXTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

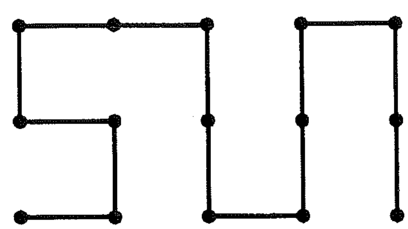
Saturday, December 3, 2005 — Examination A

Problem A1

Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another.
(For example, $23 = 9 + 8 + 6$.)

Problem A2

Let $S = \{(a,b) \mid a = 1,2,\ldots,n,\ b = 1,2,3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (*i*) $p_i \in S$, (*ii*) p_i and p_{i+1} are a unit distance apart, for $1 \le i < 3n$, (*iii*) for each $p \in S$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at (1,1) and end at $(n,1)$?
(An example of such a rook tour for $n = 5$ is depicted below.)



Problem A3

Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

SIXTY-SIXTH ANNUAL
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Saturday, December 3, 2005

Problem A4

Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Problem A5

Evaluate $\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$.

Problem A6

Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

SIXTY-SIXTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 3, 2005 — Examination B

Problem B1

Find a nonzero polynomial $P(x,y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a.
(Note: $\lfloor v \rfloor$ is the greatest integer less than or equal to v.)

Problem B2

Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Problem B3

Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Problem B4

For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$.

SIXTY-SIXTH ANNUAL
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Saturday, December 3, 2005

Examination B

Problem B5

Let $P(x_1,\ldots,x_n)$ denote a polynomial with real coefficients in the variables $x_1,\ldots,x_n$, and suppose that

(a) $$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)P(x_1,\ldots,x_n)=0 \quad \text{(identically)}$$

and that

(b) $$x_1^2+\cdots+x_n^2 \text{ divides } P(x_1,\ldots,x_n).$$

Show that $P=0$ identically.

Problem B6

Let S_n denote the set of all permutations of the numbers 1, 2, … , n. For $\pi \in S_n$, let $\sigma(\pi)=1$ if π is an even permutation and $\sigma(\pi)=-1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1}=(-1)^{n+1}\frac{n}{n+1}.$$

SIXTY-FIFTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 4, 2004

Examination A

Problem A1

Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less thant 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

Problem A2

For $i = 1,\ 2$, let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

Problem A3

Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det\begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \ge 0$. Show that u_n is an integer for all n. (By convention, 0!=1.)

Problem A4

Show that for any positive integer n there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers, $-1, 0, 1$.

Problem A5

An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$.We say that two squares, p and q, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Problem A6

Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)\,dy \right)^2 dx$$
$$\leq \left(\int_0^1 \int_0^1 f(x,y)\,dx\,dy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2\,dx\,dy.$$

SIXTY-FIFTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 4, 2004 — Examination B

Problem B1

Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\quad c_n r^2 + c_{n-1} r,\quad c_n r^3 + c_{n-1} r^2 + c_{n-2} r,\quad \ldots,\quad c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

Problem B2

Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

Problem B3

Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y) : 0 \le x \le a,\ 0 \le y \le f(x)\}$$

has perimeter k units and area k square units for some real number k.

Problem B4

Let n be a positive integer, $n \ge 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots,$ then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

Problem B5

Evaluate

$$\lim_{x\to1^-}\prod_{n=0}^{\infty}\left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

Problem B6

Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. Let $\mathcal{B}$ denote the set of positive integers b that can be written in the form $b=a-a'$ with $a\in\mathcal{A}$ and $a'\in\mathcal{A}$. Let $b_1<b_2<\cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1}-b_i$ is unbounded, then $\lim_{x\to\infty}N(x)/x=0$.

SIXTY-FOURTH ANNUAL

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 6, 2003

Examination A

Problem A1

Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$, there are four ways: $4,\ 2+2,\ 1+1+2,\ 1+1+1+1$.

Problem A2

Let $a_1,\ a_2,\ \ldots,\ a_n$ and $b_1,\ b_2,\ \ldots,\ b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \le \left((a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\right)^{1/n}.$$

Problem A3

Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

Problem A4

Suppose that a, b, c, A, B, C are real numbers, $a \ne 0$ and $A \ne 0$, such that

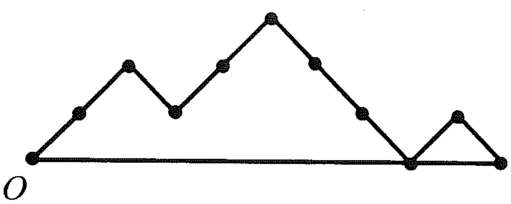
$$|ax^2 + bx + c| \le |Ax^2 + Bx + C|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \le |B^2 - 4AC|.$$

Problem A5

A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

Problem A6

For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S,\ \ s_2 \in S,\ \ s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

SIXTY-FOURTH ANNUAL

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 6, 2003

Examination B

Problem B1

Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1+xy+x^2y^2=a(x)c(y)+b(x)d(y)$$

holds identically?

Problem B2

Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries and continue until the final sequence produced consists of a single number x_n. Show that $x_n<\frac{2}{n}$.

Problem B3

Show that for each positive integer n,

$$n!=\prod_{i=1}^{n}\operatorname{lcm}\{1,2,\ldots,\lfloor n/i\rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x\rfloor$ denotes the greatest integer $\leq x$.)

Problem B4

Let $f(z)=az^4+bz^3+cz^2+dz+e=a(z-r_1)(z-r_2)(z-r_3)(z-r_4)$ where a,b,c,d,e are integers, $a\neq 0$. Show that if r_1+r_2 is a rational number, and if $r_1+r_2\neq r_3+r_4$, then r_1r_2 is a rational number.

Problem B5

Let A, B and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let $a,\ b,\ c$ be the distances from P to $A,\ B,\ C$ respectively. Show that there is a triangle with side lengths $a,\ b,\ c$, and that the area of this triangle depends only on the distance from P to O.

Problem B6

Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1 \int_0^1 |f(x)+f(y)|\,dx\,dy \geq \int_0^1 |f(x)|\,dx.$$

SIXTY-THIRD ANNUAL

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 7, 2002 — Examination A

Problem A1

Let k be a fixed positive integer. The nth derivative of $\frac{1}{x^k - 1}$ has the form $\frac{P_n(x)}{(x^k - 1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Problem A2

Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Problem A3

Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Problem A4

In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Problem A5

Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots \right\}.$$

Problem A6

Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

SIXTY-THIRD ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 7, 2002 **Examination B**

Problem B1

Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Problem B2

Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

Problem B3

Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

Problem B4

An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

Problem B5

A palindrome in base b is a positive integer whose base-b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

Problem B6

Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax+by+cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

SIXTY-SECOND ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 1, 2001

Examination A

Problem A1

Consider a set S and a binary operation $*$ on S (that is, for each a, b in S, $a * b$ is in S). Assume that $(a * b) * a = b$ for all a, b in S. Prove that $a * (b * a) = b$ for all a, b in S.

Problem A2

You have coins $C_1, C_2, \ldots, C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Problem A3

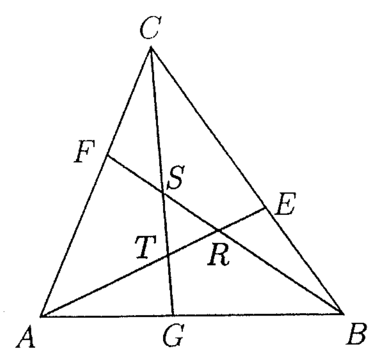
For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

Problem A4

Triangle ABC has area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of triangle RST.



Problem A5

Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001.$$

Problem A6

Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

SIXTY-SECOND ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 1, 2001

Examination B

Problem B1

Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from left to right, is

$$(k-1)n+1, \quad (k-1)n+2, \quad \ldots, \quad (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Problem B2

Find all pairs of real numbers (x, y) satisfying the system of equations

$$\frac{1}{x}+\frac{1}{2y}=(x^2+3y^2)(3x^2+y^2)$$

$$\frac{1}{x}-\frac{1}{2y}=2(y^4-x^4).$$

Problem B3

For any positive integer n let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty}\frac{2^{\langle n \rangle}+2^{-\langle n \rangle}}{2^n}.$$

Problem B4

Let S denote the set of rational numbers different from $-1, 0$ and 1. Define $f : S \to S$ by $f(x) = x - \frac{1}{x}$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \varnothing,$$

where $f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.

(Note: $f(S)$ denotes the set of all values $f(s)$ for $s \in S$.)

Problem B5

Let a and b be real numbers in the interval $(0, \frac{1}{2})$ and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

Problem B6

Assume that $(a_n)_{n \ge 1}$ is an increasing sequence of positive real numbers such that $\lim_{n \to \infty} \frac{a_n}{n} = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-i} + a_{n+i} < 2a_n \quad \text{for} \quad i = 1, 2, \ldots, n-1?$$

SIXTY-FIRST ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 2, 2000 **Examination A**

Problem A1

Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Problem A2

Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers.
[*Example*: $0 = 0^2 + 0^2, 1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

Problem A3

The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Problem A4

Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2)\, dx$$

converges.

Problem A5

Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Problem A6

Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

SIXTY-FIRST ANNUAL

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 2, 2000

Examination B

Problem B1

Let a_j, b_j, and c_j be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Problem B2

Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $\gcd(m,n)$ is the greatest common divisor of m and n.]

Problem B3

Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and $a_N \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $\frac{d^k f}{dt^k}$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \quad \text{and} \quad \lim_{k\to\infty} N_k = 2N.$$

Problem B4

Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Problem B5

Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows:

Integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n.

Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

Problem B6

Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

SIXTIETH ANNUAL

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 4, 1999

Examination A

Problem A1

Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that, for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases} -1 & \text{if} \quad x<-1 \\ 3x+2 & \text{if} \quad -1\le x\le 0 \\ -2x+2 & \text{if} \quad x>0. \end{cases}$$

Problem A2

Let $p(x)$ be a polynomial that is non-negative for all x. Prove that, for some k, there are polynomials $f_1(x), \dots, f_k(x)$ such that

$$p(x)=\sum_{j=1}^{k}(f_j(x))^2.$$

Problem A3

Consider the power series expansion

$$\frac{1}{1-2x-x^2}=\sum_{n=0}^{\infty}a_n x^n.$$

Prove that, for each integer $n\ge 0$, there is an integer m such that

$$a_n^2+a_{n+1}^2=a_m.$$

Problem A4

Sum the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m\left(n3^m+m3^n\right)}.$$

Problem A5

Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)|\, dx.$$

Problem A6

The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1,\ a_2 = 2,\ a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

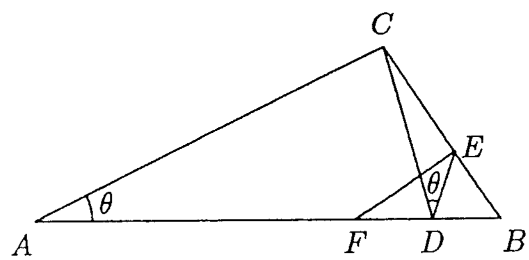
Show that, for all n, a_n is an integer multiple of n.

SIXTIETH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 4, 1999

Examination B

Problem B1

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$. [Here, $|PQ|$ denotes the length of the line segment PQ.]



Problem B2

Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots. [The roots may be either real or complex.]

Problem B3

Let $A = \{(x, y) : 0 \le x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \le \frac{m}{n} \le 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{\substack{(x,y) \to (1,1) \\ (x,y) \in A}} (1 - xy^2)(1 - x^2 y) S(x, y).$$

Problem B4

Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Problem B5

For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

Problem B6

Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n) = 1$ or $\gcd(s,n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s,t)$ is prime. [Here, $\gcd(a,b)$ denotes the greatest common divisor of a and b.]

FIFTY-NINTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 5, 1998

Examination A

Problem A1

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem A2

Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

Problem A3

Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a)\cdot f'(a)\cdot f''(a)\cdot f'''(a)\geq 0.$$

Problem A4

Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

Problem A5

Let $\mathcal{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1,\ldots,D_n$ in $\mathcal{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

Problem A6

Let $A,\ B,\ C$ denote distinct points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$(|AB|+|BC|)^2 < 8\cdot[ABC]+1$$

then $A,\ B,\ C$ are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

FIFTY-NINTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 5, 1998

Examination B

Problem B1

Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x > 0$.

Problem B2

Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

Problem B3

Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P a regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, and β are real numbers.

Problem B4

Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

Problem B5

Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111\cdots11}_{1998\text{ digits}}.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

Problem B6

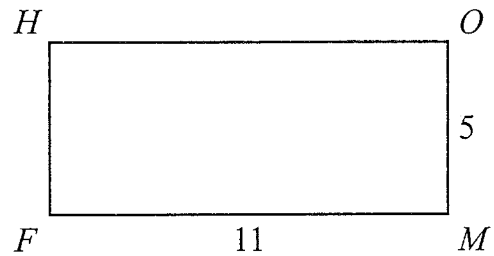
Prove that, for any integers $a,\ b,\ c,$ there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

FIFTY-EIGHTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 6, 1997 — Examination A

Problem A1

A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?



Problem A2

Players $1, 2, 3, \ldots, n$ are seated around a table and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

Problem A3

Evaluate

$$\int_0^{\infty}\left(x-\frac{x^3}{2}+\frac{x^5}{2\cdot 4}-\frac{x^7}{2\cdot 4\cdot 6}+\cdots\right)\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\cdots\right)dx.$$

Problem A4

Let G be a group with identity e and $\phi\colon G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element a in G such that $\psi(x) = a\phi(x)$ is a homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all x and y in G).

Problem A5

Let N_n denote the number of ordered n- tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

Problem A6

For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

FIFTY-EIGHTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 6, 1997 — Examination B

Problem B1

Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$S_n = \sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

(Here, $\min(a,b)$ denotes the minimum of a and b.)

Problem B2

Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -x\,g(x) f'(x),$$

where $g(x) \ge 0$ for all real x. Prove that $|f(x)|$ is bounded.

Problem B3

For each positive integer n write the sum $\sum_{m=1}^{n} \frac{1}{m}$ in the form $\frac{p_n}{q_n}$ where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

Problem B4

Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all $k \ge 0$,

$$0 \le \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \le 1.$$

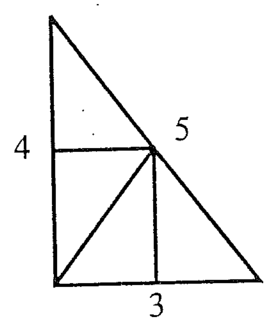
Problem B5

Prove that for $n \geq 2$,

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}n \equiv 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}n-1 \pmod{n}.$$

Problem B6

The dissection of the 3-4-5 triangle shown below has diameter 5/2.



Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)

FIFTY-SEVENTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 7, 1996 Examination A

Problem A1

Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed into that rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangle.

Problem A2

Let C_1 and C_2 be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exist points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

Problem A3

Suppose that each of twenty students has made a choice of anywhere from zero to six courses from a total of six courses offered. Prove or disprove: There are five students and two courses such that all five have chosen both courses or all five have chosen neither.

Problem A4

Let S be a set of ordered triples (a,b,c) of distinct elements of a finite set A. Suppose that:

1. $(a,b,c) \in S$ if and only if $(b,c,a) \in S$,
2. $(a,b,c) \in S$ if and only if $(c,b,a) \notin S$,
3. (a,b,c) and (c,d,a) are both in S if and only if (b,c,d) and (d,a,b) are both in S.

Prove that there exists a one-to-one function $g: A \to \mathbf{R}$ such that $g(a) < g(b) < g(c)$ implies $(a,b,c) \in S$. [Note: $\mathbf{R}$ is the set of real numbers.]

Problem A5

If p is a prime number greater than 3, and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

(For example, $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 7 + 21 + 35 + 35 = 2 \cdot 7^2$.)

Problem A6

Let $c \geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f\colon \mathbf{R} \to \mathbf{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbf{R}$.
[Note: $\mathbf{R}$ is the set of real numbers.]

FIFTY-SEVENTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 7, 1996

Examination B

Problem B1

Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Problem B2

Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}$$

Problem B3

Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \ge 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

Problem B4

For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: There exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

Problem B5

Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \le \Delta(T) \le 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

Problem B6

Let $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1,b_1)x^{a_1}y^{b_1}+(a_2,b_2)x^{a_2}y^{b_2}+\cdots+(a_n,b_n)x^{a_n}y^{b_n}=(0,0).$$

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A1

FIFTY-SIXTH ANNUAL

Saturday, December 2
1995

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem A1

Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Problem A2

For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

Problem A3

The number $d_1d_2\ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\ldots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1d_2\ldots d_9$ by the corresponding digit e_i $(1 \le i \le 9)$ is divisible by 7. The number $f_1f_2\ldots f_9$ is related to $e_1e_2\ldots e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7.
[For example, if $d_1d_2\ldots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

Problem A4

Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \le k-1 \qquad \text{for } k = 1, 2, \ldots, n.$$

Problem A5

Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\
\vdots \qquad & \qquad \vdots \qquad\qquad\qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} \geq 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

Problem A6

Suppose that each of n people writes down the numbers 1, 2, 3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

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FIFTY-SIXTH ANNUAL

B1

Saturday, December 2
1995

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B1

For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.
[A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

Problem B2

An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin\left(\frac{x}{a}\right)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

Problem B3

To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det\begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

Problem B4

Evaluate

$$\sqrt[8]{2207-\cfrac{1}{2207-\cfrac{1}{2207-\dots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where $a,\ b,\ c,\ d$ are integers.

Problem B6

For a positive real number α, define

$$S(\alpha)=\{\lfloor n\alpha\rfloor : n=1,2,3,\dots\}.$$

Prove that $\{1,2,3,\dots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$. [As usual, $\lfloor x\rfloor$ is the greatest integer $\le x$.]

Problem B5

A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

a. one bean from a heap, provided at least two beans are left behind in that heap, **or**

b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

FIFTY-FIFTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 3, 1994

Examination A

Problem A1

Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Problem A2

Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

Problem A3

Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

Problem A4

Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

Problem A5

Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}, \text{ with } i_1 < i_2 < \cdots < i_{1994}.$$

Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

Problem A6

Let $f_1, f_2, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathcal{F} = \left\{ f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} \mid e_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq 10 \right\}$$

$\left(f_i^0\right.$ is the identity function and $\left.f_i^1 = f_i\right)$. Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathcal{F}$ map A to itself.

FIFTY-FIFTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 3, 1994

Examination B

Problem B1

Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \ldots\}$. a is **within** n of b if $b - n \leq a \leq b + n$.)

Problem B2

For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

Problem B3

Find the set of all real numbers k with the following property:

For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

Problem B4

For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\lim_{n \to \infty} d_n = \infty$.

Problem B5

For any real number α, define the function f_α by $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

($\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and $f_\alpha^k = f_\alpha \circ \cdots \circ f_\alpha$ is the k-fold composition of f_α.)

Problem B6

For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \le a, b, c, d \le 99$,

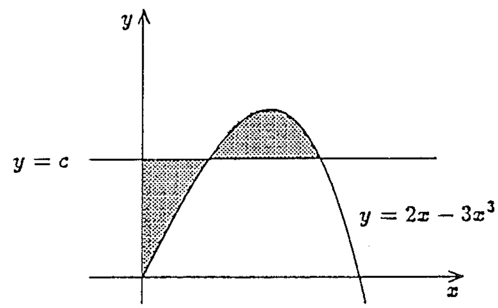
$$n_a + n_b \equiv n_c + n_d \pmod{10100} \text{ implies } \{a, b\} = \{c, d\}.$$

FIFTY-FOURTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 4, 1993 Examination A

Problem A1

The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



Problem A2

Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1, \quad \text{for } n = 1, 2, 3, \ldots.$$

Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

Problem A3

Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f\colon \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

Problem A4

Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

Problem A5

Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

is a rational number.

Problem A6

The infinite sequence of 2's and 3's

$$2,3,3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,3,2,\ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

FIFTY-FOURTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 4, 1993 **Examination B**

Problem B1

Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

Problem B2

Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. If we assume optimal strategy by both A and B, what is the probability that A wins?

Problem B3

Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

Problem B4

The function $K(x, y)$ is positive and continuous for $0 \le x \le 1$, $0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all x, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x,y)\,dy = g(x) \qquad \text{and} \qquad \int_0^1 g(y)K(x,y)\,dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

Problem B5

Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

Problem B6

Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$, and replace them with $2x$ and $y - x$.

1989 Putnam Exam

The 1989 William Lowell Putnam Mathematical Competition was administered on Saturday, 2 December 1989. The following are the questions from the exam.

Problem A-1. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

Problem A-2. Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where a and b are positive.

Problem A-3. Prove that if $11z^{10} + 10iz^9 + 10iz - 11 = 0$, then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

Problem A-4. If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an homest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

Problem A-5. Let m be a positive integer and let $\mathcal{G}$ be a regular $(2m+1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following property. For any point p inside $\mathcal{G}$ there are two distinct vertices v_1 and v_2 of $\mathcal{G}$ such that

$$\big||p - v_1| - |p - v_2|\big| < \frac{1}{m} - \frac{A}{m^3}.$$

Here $|s - t|$ denotes the distance between the points s and t.

Problem A-6. Let $\alpha = 1 - a_1x + a_2x^2 + \ldots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block,} \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$ and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + \alpha x + 1 = 0$.

Problem B-1. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability tha the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{c}$, where a, b, c, d are integers.

Problem B-2. Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

Problem B-3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\, dx$$

(sometimes called the n-th moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

Problem B-4. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

Problem B-5. Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $\frac{s_1 - s_2}{d}$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

Problem B-6. Let $(x_1, x_2, \ldots, x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\, dt$, where P is a polynomial of degree n, independent of f, with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.