Math 5351

Homework 3

- p. 37: #1, 2, 4, 5
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#2, p.17 Write the equation of an ellipse, hyperbola, and parabola in complex form.

Recall that a quadratic equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ (A,B,C,D,E,F real numbers with A,B,C not all zero) has a graph that is:

(1) an ellipse if and only if $B^2-4AC<0$;

(2) an hyperbola if and only if $B^2-4AC>0$;

(3) a parabola if and only if $B^2-4AC=0$.

Substituting $x=\dfrac{z+\bar z}{2}$ and $y=\dfrac{z-\bar z}{2i}$ in the quadratic equation yields

$$A\left(\frac{z+\bar z}{2}\right)^2+B\left(\frac{z+\bar z}{2}\right)\left(\frac{z-\bar z}{2i}\right)+C\left(\frac{z-\bar z}{2i}\right)^2+D\left(\frac{z+\bar z}{2}\right)+E\left(\frac{z-\bar z}{2i}\right)+F=0.$$

Multiplying through by 4 and expanding squares and products gives

$$A(z^2+2z\bar z+\bar z^2)-iB(z^2-\bar z^2)-C(z^2-2z\bar z+\bar z^2)+2D(z+\bar z)-2iE(z-\bar z)+4F=0.$$

Collecting "like terms" in z we have

$$\alpha z^2+\beta z\bar z+\gamma\bar z^2+\delta z+\varepsilon\bar z+\varphi=0$$

where $\alpha=A-iB-C=\bar\gamma$, $\beta=2(A+C)$, $\delta=2(D-iE)=\bar\varepsilon$, $\varphi=4F$.

Therefore a quadratic equation in complex form

(*) $$\alpha z^2+\beta z\bar z+\bar\alpha\bar z^2+\delta z+\bar\delta\bar z+\varphi=0$$

has a graph that is

(1) an ellipse if and only if $|\alpha|^2-\frac14|\beta|^2=B^2-4AC<0$;

(2) an hyperbola if and only if $|\alpha|^2-\frac14|\beta|^2=B^2-4AC>0$;

(3) a parabola if and only if $|\alpha|^2-\frac14|\beta|^2=B^2-4AC=0$.

Note: Any nonzero complex multiple of (*) will be an equation of the same types (1), (2), (3).

#1, p.37 Prove that a convergent sequence is bounded.

Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit $L = \lim_{n\to\infty} a_n$.

Then there exists an index $n_0 \geq 1$ such that $|a_n - L| < 1$ for all $n > n_0$. We claim that all terms in the sequence $\{a_n\}_{n=1}^{\infty}$ satisfy $|a_n| \leq B$ where

$$B = \max\{|a_1|, |a_2|, \dots, |a_{n_0}|, |L|+1\}.$$

The inequality $|a_n| \leq B$ is clear if $1 \leq n \leq n_0$. Suppose $n > n_0$. Then the triangle inequality implies $|a_n| \leq |a_n - L| + |L|$. But $|a_n - L| < 1$ since $n > n_0$ so $|a_n| < 1 + |L|$.

#2, p.37   If $\lim_{n\to\infty} z_n = A$, prove that $\lim_{n\to\infty} \frac{1}{n}(z_1+z_2+\dots+z_n) = A$.

It suffices to prove that $\lim_{n\to\infty}\left[\frac{1}{n}(z_1+z_2+\dots+z_n) - A\right] = 0$.

Let $\varepsilon>0$ be given. Since $A = \lim_{n\to\infty} z_n$, there exists an index $n_1 \geq 1$ such that

$$|z_n - A| < \frac{\varepsilon}{3} \quad \text{for all } n \geq n_1.$$

For $n > n_1$, observe that

$$\frac{1}{n}(z_1+z_2+\dots+z_n) - A = \frac{1}{n}(z_1+z_2+\dots+z_{n_1}) + \frac{1}{n}\Big(\overbrace{(z_{n_1+1}-A)+\dots+(z_n-A)}^{n-n_1 \text{ terms}}\Big) - \frac{n_1}{n}A.$$

Since $\lim_{n\to\infty} \frac{1}{n}(z_1+\dots+z_{n_1}) = 0 = \lim_{n\to\infty}\frac{n_1 A}{n}$ there exist indices $n_2 \geq n_1$ and $n_3 \geq n_1$ such that

$$\left|\frac{1}{n}(z_1+\dots+z_{n_1})\right| < \frac{\varepsilon}{3} \quad \text{for all } n \geq n_2$$

and

$$\left|\frac{n_1 A}{n}\right| < \frac{\varepsilon}{3} \quad \text{for all } n \geq n_3.$$

Consequently, if $n \geq \max\{n_1, n_2, n_3\}$ then

$$\left|\frac{1}{n}(z_1+z_2+\dots+z_n) - A\right| \leq \left|\frac{1}{n}(z_1+\dots+z_{n_1})\right| + \frac{1}{n}\big(|z_{n_1+1}-A|+\dots+|z_n-A|\big) + \left|\frac{n_1 A}{n}\right|$$

#2, p. 37 (cont.)

$$\leq \frac{\varepsilon}{3} + \frac{1}{n}(n-n_1)\frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon .$$

This shows that $\lim_{n\to\infty} \left\{ \frac{1}{n}(z_1+z_2+\cdots+z_n) - A \right\} = 0.$

#4, p. 37 Discuss completely the convergence and uniform convergence of the sequence $\{nz^n\}_{n=1}^{\infty}$.

Let $f_n(z) = nz^n$ for $n=1,2,3,\dots$ and $z \in \mathbb{C}$. If $|z| \geq 1$ then $|f_n(z)| = |nz^n| = n|z|^n \geq n$, so $\lim_{n\to\infty} f_n(z)$ does not exist if $|z| \geq 1$. Clearly $f_n(0) = 0$ for all $n \geq 1$ so $\lim_{n\to\infty} f_n(0) = 0$. Suppose $0 < |z| < 1$. Then writing $p = \frac{1}{|z|} - 1$, note that $0 < p < \infty$ so

$$|nz^n| = n|z|^n = \frac{n}{\left(\frac{1}{|z|}\right)^n} = \frac{n}{(1+p)^n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

(See the lemma at the end of this problem.) Therefore the sequence of functions $\{f_n(z)\}_{n=1}^{\infty} = \{nz^n\}_{n=1}^{\infty}$ converges (pointwise) to the zero function in the open unit disk

$$D(0;1) = \{ z \in \mathbb{C} : |z| < 1 \}$$

and diverges on the circumference or outside this unit disk.

Careful examination of this argument shows that for each $r \in (0,1)$, the convergence of $\{f_n\}_{n=1}^{\infty}$ to the zero function is uniform on the closed disk

$$\overline{D(0;r)} = \{ z \in \mathbb{C} : |z| \le r \}.$$

To see this, let r satisfy $0 < r < 1$ and set $p = \frac{1}{r} - 1 > 0$.

Since $\lim_{n\to\infty} \frac{n}{(1+p)^n} = 0$, given $\varepsilon > 0$ there corresponds

an index $n_0 \ge 1$ such that $\frac{n}{(1+p)^n} < \varepsilon$ for all $n \ge n_0$.

If $z \in \overline{D(0;r)}$ then

$$|f_n(z)| = |n z^n| \le n r^n = \frac{n}{(1+p)^n} < \varepsilon$$

for all $n \ge n_0$; i.e. $f_n \to 0$ uniformly on $\overline{D(0;r)}$.

However, $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to the zero function on the open unit disk $D(0;1)$. To see this, observe that the numerical sequence $\left\{\left(1-\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ is increasing, so for all $n \ge 2$ we have

$$f_n\left(1-\tfrac{1}{n}\right) = n\left(1-\tfrac{1}{n}\right)^n \ge n\left(1-\tfrac{1}{2}\right)^2 = \frac{n}{4}.$$

Thus there is no index $n_0 \ge 1$ such that $|f_n(z)| < 1$ for all

$n \ge n_0$ and all $z \in D(0;1)$.

Appendix to #4, p. 37:

Lemma: If $p > 0$ and α is real then $\lim_{n \to \infty} \frac{n^\alpha}{(1+p)^n} = 0$.

Proof: Choose and fix an integer k such that $k > \max\{\alpha, 0\}$. Then for $n > 2k$ we have

$$(1+p)^n = \sum_{j=0}^{n} \binom{n}{j} p^j > \binom{n}{k} p^k = \frac{n(n-1)\ldots(n-k+1)}{k!} p^k > \left(\frac{n}{2}\right)^k \cdot \frac{p^k}{k!}.$$

Consequently, if $n > 2k$ then

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{n^\alpha}{\left(\frac{n}{2}\right)^k \cdot \frac{p^k}{k!}} = \left(\frac{2^k k!}{p^k}\right) n^{\alpha - k}.$$

Since $\alpha - k < 0$, $\lim_{n \to \infty} n^{\alpha - k} = 0$. The desired conclusion follows.

#5, p.37 Discuss uniform convergence of the series

$$(*) \qquad \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

for real values of x.

Observe first that if $f(y) = \frac{y}{1+y^2}$ for $y \in \mathbb{R}$ then $f'(y) = \frac{(1+y^2)(1) - y(2y)}{(1+y^2)^2}$ $= \frac{1-y^2}{(1+y^2)^2}$. Therefore $f \searrow$ on $(-\infty,-1)$, $f \nearrow$ on $(-1,1)$, and $f \searrow$ on $(1,\infty)$. Since $\lim_{y\to\pm\infty} f(y) = 0$, it follows that

$$(**) \qquad \max\left\{ \left| \frac{y}{1+y^2} \right| : y \in \mathbb{R} \right\} = \max\left\{ \frac{y}{1+y^2} : y \geq 0 \right\} = f(1) = \tfrac{1}{2}.$$

Now let n be a positive integer and $x \in \mathbb{R}$. We have by $(**)$

$$(***) \qquad \left| \frac{x}{n(1+nx^2)} \right| = \left| \frac{x\sqrt{n}}{n\sqrt{n}(1+nx^2)} \right| = \frac{1}{n^{3/2}} \left| \frac{x\sqrt{n}}{1+(x\sqrt{n})^2} \right| \leq \frac{1/2}{n^{3/2}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges – it is a p-series with $p = 3/2 > 1$ – it follows from $(***)$ that the series $(*)$ converges absolutely and uniformly on $\mathbb{R}$ by the Weierstrass M-test (cf. p. 37).

#1, p. 41 Expand $(1-z)^{-m}$, m a positive integer, in powers of z.

Let $f(z)=(1-z)^{-m}$ where $|z|<1$. Assuming this analytic function has a power series expansion in powers of z, it must be of the form

$$f(z)= f(0)+ f'(0)z + \frac{f''(0)}{2!}z^2 + \dots + \frac{f^{(n)}(0)}{n!}z^n + \dots$$

We compute:

$$f(0)=(1-0)^{-m}=1,$$

$$f'(z)=-m(1-z)^{-(m+1)}(-1) = m(1-z)^{-(m+1)} \quad \text{so}$$

$$f'(0)= m,$$

$$f''(z)= -m(m+1)(1-z)^{-(m+2)}(-1) = m(m+1)(1-z)^{-(m+2)} \quad \text{so}$$

$$f''(0)= m(m+1),$$

and, by a mathematical induction argument,

$$f^{(n)}(z)= m(m+1)\dots(m+n-1)(1-z)^{-(m+n)} \quad \text{so}$$

$$f^{(n)}(0) = m(m+1)\dots(m+n-1) = \frac{(m+n-1)!}{(m-1)!}$$

for all integers $n \geq 1$.

Consequently,

$$f(z) = 1 + mz + \frac{m(m+1)}{2!} z^2 + \cdots + \frac{(m+n-1)!}{n!(m-1)!} z^n + \cdots$$

or $$(1-z)^{-m} = 1 + \sum_{n=1}^{\infty} \binom{m+n-1}{n} z^n ,$$

where we have used the familiar notation

$$\binom{\ell}{k} = \frac{\ell!}{k!(\ell-k)!}$$

for the binomial coefficent "ℓ choose k".

Note: The coefficients $a_n = \binom{m+n-1}{n}$ in the expansion of $f(z) = (1-z)^{-m}$ in powers of z satisfy

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{m+n} = 1$$

so the power series representing f converges for $|z| < 1$ (cf. #7, p. 41).

#2, p.41 Expand $f(z) = \frac{2z+3}{z+1}$ in powers of $z-1$. What is the radius of convergence?

$$f(z) = \frac{2z+3}{z+1} = \frac{2(z+1)+1}{z+1} = 2 + \frac{1}{z+1} = 2 + \frac{1}{2+(z-1)} = 2 + \frac{1}{2} \cdot \frac{1}{1+\left(\frac{z-1}{2}\right)}.$$

Using the geometric series $\frac{1}{1+r} = \sum_{n=0}^{\infty} (-r)^n$ for all $|r|<1$, we see that

$$f(z) = 2 + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{(z-1)}{2}\right)^n = \frac{5}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n (z-1)^n}{2^{n+1}} \quad \text{for all } \left|\frac{z-1}{2}\right| < 1.$$

Hence the radius of convergence is $R = 2$.

Check: $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n \overset{\text{(See below)}}{=} \frac{5}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-1)^n$

$f(z) = 2 + (z+1)^{-1}$ so $f(1) = \frac{5}{2}$

$f'(z) = -(z+1)^{-2}$ so $f'(1) = -\frac{1}{2^2}$

$f''(z) = 2(z+1)^{-3}$ so $f''(1) = 2 \cdot 2^{-3}$

$f'''(z) = -6(z+1)^{-4}$ so $f'''(1) = -6 \cdot 2^{-4}$

$\vdots \qquad \vdots$

$f^{(n)}(z) = (-1)^n n!\,(z+1)^{-(n+1)}$ so $f^{(n)}(1) = (-1)^n n!\, 2^{-(n+1)}$

#3, p.41.

(cf. #7, p.41)

$\sum_{n=1}^{\infty} n^p z^n$ has radius of convergence $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^p\right| = 1$.

$\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} =$

$\lim_{n\to\infty} n+1 = \infty$.

$\sum_{n=0}^{\infty} n! z^n$ has radius of convergence $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} =$

$\lim_{n\to\infty} \frac{1}{n+1} = 0$.

If $|q|<1$ then $\sum_{n=0}^{\infty} q^{n^2} z^n$ has radius of convergence R satisfying

$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \left(|q|^{n^2}\right)^{1/n} = \limsup_{n\to\infty} |q|^n = 0$.

Thus $R = \infty$.

$\sum_{n=0}^{\infty} z^{n!}$ has radius of convergence R satisfying $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Since $a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some integer } k \geq 0, \\ 0 & \text{otherwise,} \end{cases}$

it follows that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$. Hence $R = 1$.

#4, p.41 If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^{2n}$? of $\sum_{n=0}^{\infty} a_n^2 z^n$?

Let R denote the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$. Then

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

The radius of convergence ρ of $\sum_{n=0}^{\infty} a_n z^{2n} = \sum_{k=0}^{\infty} b_k z^k$ satisfies

$$\frac{1}{\rho} = \limsup_{k\to\infty} \sqrt[k]{|b_k|} \quad \text{where} \quad b_k = \begin{cases} a_n & \text{if } k=2n \text{ for some integer } n, \\ 0 & \text{if } k=2n+1 \text{ for some integer } n. \end{cases}$$

Therefore

$$\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[2n]{|a_n|} = \limsup_{n\to\infty} \left(\sqrt[n]{|a_n|}\right)^{1/2} = \left(\frac{1}{R}\right)^{1/2}.$$

Consequently the radius of convergence ρ of $\sum_{n=0}^{\infty} a_n z^{2n}$ is $\rho = \sqrt{R}$.

The radius of convergence r of $\sum_{n=0}^{\infty} a_n^2 z^n$ satisfies

$$\frac{1}{r} = \limsup_{n\to\infty} \sqrt[n]{|a_n|^2} = \limsup_{n\to\infty} \left(\sqrt[n]{|a_n|}\right)^2 = \frac{1}{R^2},$$

so $r = R^2$.

#7, p.41 Let $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exist (in the extended real number sense).

Then the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence R.

Case 1: Suppose $0 < R < \infty$. Let $\varepsilon \in (0, R)$. Then there exists an index $N \geq 1$ such that $R-\varepsilon < \left|\frac{a_n}{a_{n+1}}\right| < R+\varepsilon$ for all $n \geq N$.

Then for every integer $m \geq 0$ we have

$$(R-\varepsilon)^{m+1} \leq \left| \frac{a_N}{\cancel{a_{N+1}}} \cdot \frac{\cancel{a_{N+1}}}{\cancel{a_{N+2}}} \cdot \ldots \cdot \frac{\cancel{a_{N+m}}}{a_{N+m+1}} \right| \leq (R+\varepsilon)^{m+1}$$

or equivalently

$$\frac{|a_N|}{(R+\varepsilon)^{m+1}} \leq |a_{N+m+1}| \leq \frac{|a_N|}{(R-\varepsilon)^{m+1}}.$$

Consequently

$$(*) \qquad \frac{|a_N|^{\frac{1}{N+m+1}}}{(R+\varepsilon)^{\frac{m+1}{N+m+1}}} \leq \sqrt[N+m+1]{|a_{N+m+1}|} \leq \frac{|a_N|^{\frac{1}{N+m+1}}}{(R-\varepsilon)^{\frac{m+1}{N+m+1}}}$$

for all integers $m \geq 0$. But if λ is a positive real number then

$$\lim_{\alpha\to 1} \lambda^\alpha = \lim_{\alpha \to 1} e^{\alpha \ln \lambda} = e^{\ln\lambda} = \lambda$$

and

$$\lim_{\alpha\to 0^+} \lambda^\alpha = \lim_{\alpha\to 0^+} e^{\alpha\ln\lambda} = e^0 = 1.$$

#7, p. 41 (cont.) It follows that

$$\lim_{m\to\infty} |a_N|^{\frac{1}{N+m+1}} = 1, \quad \lim_{m\to\infty} (R\pm\varepsilon)^{\frac{m+1}{N+m+1}} = R\pm\varepsilon.$$

Applying this to (*) we obtain

$$\frac{1}{R+\varepsilon} = \lim_{m\to\infty} \frac{|a_N|^{\frac{1}{N+m+1}}}{(R+\varepsilon)^{\frac{m+1}{N+m+1}}} \le \liminf_{m\to\infty} \sqrt[N+m+1]{|a_{N+m+1}|}$$

$$\le \limsup_{m\to\infty} \sqrt[N+m+1]{|a_{N+m+1}|}$$

$$\le \lim_{m\to\infty} \frac{|a_N|^{\frac{1}{N+m+1}}}{(R-\varepsilon)^{\frac{m+1}{N+m+1}}} = \frac{1}{R-\varepsilon}.$$

But $\varepsilon>0$ is arbitrary so it follows that $\lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists and is equal to $\frac{1}{R}$. Consequently, the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R.

Case 2: Suppose $R=0$. Then the argument of Case 1 shows that to each $\varepsilon>0$ there corresponds an integer $N\ge 1$ such that

$$\frac{|a_N|^{\frac{1}{N+m+1}}}{\varepsilon^{\frac{m+1}{N+m+1}}} \le \sqrt[N+m+1]{|a_{N+m+1}|} \qquad (m=0,1,2,\ldots)$$

and it follows that $\frac{1}{\varepsilon} \le \limsup_{k\to\infty} \sqrt[k]{|a_k|}$. But

#7, p.41 (cont.) $\varepsilon > 0$ is arbitrary so $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \infty$ and hence $R = 0$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Case 3: Suppose $R = \infty$. Then to each $\overbrace{\rho}^{\text{real number}} > 0$ there corresponds an index $N \geq 1$ such that $\rho < \left|\frac{a_n}{a_{n+1}}\right|$ for all $n \geq N$. The argument of Case 1 shows that

$$\sqrt[N+m+1]{|a_{N+m+1}|} \leq \frac{|a_N|^{\frac{1}{N+m+1}}}{\rho^{\frac{m+1}{N+m+1}}} \qquad (m = 0, 1, 2, \ldots)$$

and it follows that

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} \leq \frac{1}{\rho}.$$

But $\rho > 0$ was arbitrary so $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 0$; thus $R = \infty$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

#8, p.41 For what values of z is $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ convergent?

Recall that the geometric series $\sum_{n=0}^{\infty} w^n$ is convergent if and only if $|w|<1$. Therefore $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ is convergent if and only if $\left|\frac{z}{z+1}\right| < 1$, or equivalently, $|z|^2 < |z+1|^2$. But

$|z|^2 = |x+iy|^2 = x^2+y^2$ and

$|z+1|^2 = |(1+x)+iy|^2 = (1+x)^2+y^2 = 1+2x+x^2+y^2$. Therefore $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ is convergent if and only if $x^2+y^2 < 1+2x+x^2+y^2$. But this is equivalent to $0 < 1+2x$ or $-\frac{1}{2} < x$. That is, the series $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ is convergent if and only if z lies in the open half-plane $\mathrm{Re}(z) > -\frac{1}{2}$.