Math 5351

Homework 5

- Lecture Problem 1
- p. 108: # 1, 2, 3, 4, 5, 7
Lecture Problem 1

(a) Show that $f(z) = \sim\log(1 + z)$ is analytic in the region

\[ R = \mathbb{C} \setminus \{ z = x + iy : x \leq -1 \text{ and } y = 0 \} \]

with derivative $f'(z) = \frac{1}{1+z}$ for $z \in R$.

(b) Show that the function $g(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$ is analytic in the open disk $D(0; 1) = \{ z \in \mathbb{C} : |z| < 1 \}$ and $g(z) = \frac{1}{1+z}$ if $z \in D(0; 1)$.

(c) Use (a) and (b) to show that $\sim\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$ if $z \in D(0; 1)$.

(d) Use (c) to show that $-\sim\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ if $z \in D(0; 1)$.

(e) Use (d), continuity of the principal logarithm function, and Abel's Limit Theorem (Theorem 3, p.41) to conclude that

\[ -\sim\log(1-e^{i\theta}) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} = \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} \]

for all $0 < \theta < 2\pi$. (Hint: What have we shown about the sequence of partial sums of the series $\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$ ?)

(f) Use $\sim\log(w) = \ln|w| + i \sim\arg(w)$ and (e) to show that

\[ \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} = \ln\left(\frac{1}{2\sin(\theta/2)}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \frac{\pi - \theta}{2} \]

for all $0 < \theta < 2\pi$.

(g) Find the sums of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.
Part(a): \( g(w) = \overline{\log w} \) is analytic in the region

\[ \mathbb{C} \setminus \{ w = u + iv : u \leq 0 \text{ and } v = 0 \}, \]

with derivative \( g'(w) = \frac{1}{w} \). The function \( w = h(z) = 1 + z \) is analytic in the entire complex plane with derivative \( h'(z) = 1 \) and maps \( \mathbb{C} \setminus \{ z = x + iy : x \leq -1 \text{ and } y = 0 \} \) onto \( \mathbb{C} \setminus \{ w = u + iv : u \leq 0 \text{ and } v = 0 \} \). Consequently the composition \( f(z) = g(h(z)) \) is analytic in the region

\[ R = \mathbb{C} \setminus \{ z = x + iy : x \leq -1 \text{ and } y = 0 \}, \]

with derivative

\[ f'(z) = g'(h(z)) h'(z) = \frac{1}{1 + z}. \]

Part(b): The function \( g(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \), being a power series with radius of convergence

\[ R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n-1} z^n}{(-1)^n/n+1} \right| = \lim_{n \to \infty} \frac{n+1}{n} = 1, \]

is analytic in its disk of convergence \( D(0; 1) \). Furthermore, its derivative can be obtained through term-by-term differentiation:

\[ g'(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} = \sum_{n=1}^{\infty} (-z)^{n-1} = \frac{1}{1 - (-z)} = \frac{1}{1 + z}, \]

the third equality following from a geometric series, convergent for \( |z| < 1 \).
Part (c): Consider the function $F$ defined in $D(0; 1)$ by

$$ F(z) = \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right) \frac{z^n}{n} . $$

Since $z \mapsto \log(1+z)$ is analytic in $\mathbb{C} \setminus \{ z=x+iy : x \leq -1 \text{ and } y=0 \}$ by part (a) and $z \mapsto \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \frac{z^n}{n}}{n}$ is analytic in $D(0; 1)$ by part (b), it follows that $F$ is analytic in $D(0; 1)$ with derivative

$$ F'(z) = \frac{1}{1+z} - \frac{1}{1+z} = 0 $$

for $|z| < 1$. Consequently $F$ is a constant function in $D(0; 1)$. But clearly $F(0) = 0$. Hence $F(z) = 0$ for all $z$ in $D(0; 1)$; i.e.

$$ \log(1+z) = \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right) \frac{z^n}{n} \quad (z \in D(0; 1)). $$

Part (d): Notice that $z$ belongs to $D(0; 1)$ if and only if $-z$ belongs to $D(0; 1)$. Therefore, replacing $z$ by $-z$ in the identity from part (c), we obtain

$$ \log(1-z) = \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right) \frac{z^n}{n} = - \sum_{n=1}^{\infty} \frac{z^n}{n} $$

for all $z \in D(0; 1)$. This is equivalent to the desired identity.

Part (e): We will use the following version of Abel's limit theorem

(Theorem 3 on page 41):
If \( \sum_{n=0}^{\infty} a_n e^{i\theta} \) converges for some \( \theta \in [0, 2\pi] \) then \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) tends to \( f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{i\theta} \) as \( z \) approaches \( e^{i\theta} \) from inside the unit disk in such a way that \( \frac{|z|}{|e^{i\theta} - z|} \) remains bounded.

Recall that we have shown (the sequence of partial sums of) the series \( \sum_{n=1}^{\infty} \frac{e^{i\theta}}{n} \) converges for all \( \theta \in (0, 2\pi) \). For all such \( \theta \) we have

\[
\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \sum_{n=1}^{\infty} \frac{e^{i\theta}}{n}
\]

\[
= \lim_{r \to 1^-} \sum_{n=1}^{\infty} \frac{(re^{i\theta})^n}{n} \quad \text{(by Abel's limit theorem)}
\]

\[
= \lim_{r \to 1^-} -\log(1-re^{i\theta}) \quad \text{(by part (d))}
\]

\[
= -\log(1-e^{i\theta}) \quad \text{(by continuity of } w \mapsto \log(w) \text{ in } \mathbb{C} \setminus \{w=\text{unit: } u<0, v=0\})
\]

Part (f): If \( 0 < \theta < 2\pi \) then

\[
\log(1-e^{i\theta}) = \ln|1-e^{i\theta}| + i \arg(1-e^{i\theta}).
\]

Note that \( 1-e^{i\theta} = 1-\cos\theta - is\sin\theta \) so \( |1-e^{i\theta}| = \sqrt{(1-\cos\theta)^2 + (-\sin\theta)^2} = \sqrt{1-2\cos\theta + \cos^2\theta + \sin^2\theta} = \sqrt{2-2\cos\theta} = \sqrt{4\sin^2(\theta/2)} = 2\sin(\theta/2) \).
Also $\widehat{\text{Arg}}(1 - e^{i\theta}) = y$ if and only if $e^{iy} = \frac{1 - e^{i\theta}}{|1 - e^{i\theta}|}$

But $\frac{1 - e^{i\theta}}{|1 - e^{i\theta}|} = \frac{(1 - \cos(\theta)) - i\sin(\theta)}{2\sin(\theta/2)} = \frac{2\sin^2(\theta/2) - i2\sin(\theta/2)\cos(\theta/2)}{2\sin(\theta/2)}$

$= \sin(\theta/2) - i\cos(\theta/2) = -i(\cos(\theta/2) + i\sin(\theta/2)) = e^{\frac{-i\pi}{2}}e^{\frac{i\theta}{2}} = e^{\frac{i(\theta - \pi)}{2}}$.

Observe that $0 < \theta < 2\pi$ implies $-\frac{\pi}{2} < \frac{\theta - \pi}{2} < \frac{\pi}{2}$. Therefore

$\widehat{\text{Arg}}(1 - e^{i\theta}) = y = \frac{\theta - \pi}{2}$.

Consequently, for all $\theta \in (0, 2\pi)$,

$-\widehat{\log}(1 - e^{i\theta}) = -\ln|1 - e^{i\theta}| = i\widehat{\text{Arg}}(1 - e^{i\theta})$

$= -\ln(2\sin(\theta/2)) - i\left(\frac{\theta - \pi}{2}\right)$

$= \ln\left(\frac{1}{2\sin(\theta/2)}\right) + i\left(\frac{\pi - \theta}{2}\right)$.

Comparing this last identity with the identity in part (e), we conclude that

$$\sum_{n=1}^{\infty} \frac{\cos(\theta)}{n} = \ln\left(\frac{1}{2\sin(\theta/2)}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin(\theta)}{n} = \frac{\pi - \theta}{2}$$

for all $0 < \theta < 2\pi$.

Part (g): Taking $\theta = \pi$ in the identity $\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} = \ln\left(\frac{1}{2\sin(\theta/2)}\right)$

yields $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{2}\right) = -\ln(2)$.
We take $\theta = \frac{\pi}{2}$ in the identity
\[ \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \frac{\pi - \theta}{2} \quad (0 < \theta < 2\pi) \]
to obtain

\[ \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \]

or

\[ \left[ \frac{\pi}{4} \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \]
Compute $\int x \, dz$, for the positive sense of the circle, $|z|=r$

in two ways: first, by use of a parameter, and second, by observing that $x = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}(z + \frac{z^2}{|z|^2})$ on the circle.

**First way:** We parametrize the circle $|z|=r$ by $z(\theta) = re^{i\theta}$ ($0 \leq \theta \leq 2\pi$) so $dz = ire^{i\theta} \, d\theta$. Hence

$$\int x \, dz = \int_{0}^{2\pi} r(\cos \theta)ire^{i\theta} \, d\theta = ir \int_{0}^{2\pi} (\cos \theta + i \cos \theta \sin \theta) \, d\theta$$

$$= ir \left[ \int_{0}^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta + i \int_{0}^{2\pi} \sin \theta \, d\theta \right] = \frac{ir}{2} \left[ \pi + \frac{i \sin 2\pi}{2} \right] = \frac{ir^2 \pi}{2}.$$

**Second way:** Substituting $x = \frac{1}{2}(z + \frac{z^2}{|z|^2})$ for $|z|=r$ and using the fact that $\int_{|z|=r} zdz = \left\{ \begin{array}{ll} 2\pi i & \text{if } m=-1, \\ 0 & \text{if } m \neq -1, \end{array} \right.$ we have

$$\int x \, dz = \int_{|z|=r} \frac{1}{2}(z + \frac{z^2}{2}) \, dz = \frac{1}{2} \int_{|z|=r} zdz + \frac{r^2}{2} \int_{\|z\|=r} \frac{dz}{2} = 0 + \frac{r^2}{2} (2\pi i)$$

$$= \frac{ir^2 \pi}{2}.$$
If $P = P(z)$ is a polynomial and $C$ denotes the circle $|z-a| = R$, what is the value of $\int_C P(z) \overline{dz}$?

(Answer: $-2\pi i R^2 P'(a)$.)

By definition (cf. p. 103), $\int_C P(z) \overline{dz} = \int_C \overline{P(z)} dz$. Now let $P$ be a polynomial function of degree $n \geq 0$ and expand $P$ in a (finite) Taylor series about $a$:

$$P(z) = \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!} (z-a)^k.$$

Since the circle $C: |z-a| = R$ has parametrization $z(\theta) = a + Re^{i\theta}$ ($0 \leq \theta \leq 2\pi$) with $dz = iRe^{i\theta} d\theta$,

$$\int_C \overline{P(z)} dz = \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!} \int_0^{2\pi} \overline{(Re^{i\theta})^k} iRe^{i\theta} d\theta$$

$$= \sum_{k=0}^{n} iR^{k+1} \frac{P^{(k)}(a)}{k!} \int_0^{2\pi} e^{i(k-1)\theta} d\theta.$$

But $\int_0^{2\pi} e^{im\theta} d\theta = \begin{cases} 2\pi & \text{if } m = 0, \\ 0 & \text{if } m \neq 0, \end{cases}$ and therefore $\int_C \overline{P(z)} dz = 2\pi i R^2 \overline{P'(a)}$.

Consequently

$$\int_C P(z) \overline{dz} = \int_C \overline{P(z)} dz = 2\pi i R^2 \overline{P'(a)} = -2\pi i R^2 P'(a).$$