Notes on the Great Theorems

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. . . ignorance of the roots of the subject has its price - no one denies that modern formulations are clear elegant and precise; it's just that it's impossible to comprehend how anyone ever thought of them.

-M. Spivak

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Acknowledgement

Sources for the material are given in the bibliography, but I want to especially express my thanks to Professor William Dunham of Hanover College, whose article in the *American Mathematical Monthly* crystallized my concept of this course, and who graciously provided a copy of the *Mathematics Supplement* that he developed for his own great theorems course. Several sections here are modifications, to varying degrees, of sections from his *Mathematics Supplement*. (Added 2011: Dunham's *Mathematics Supplement* later became his book, *Journey Through Genius*.)

Introduction

The quote from M. Spivak could well illustrate one of the reasons that, despite the increasing prominence of mathematics in today's world, most people just plain don't like it. Mathematicians must accept a large part of the responsibility for this state of affairs, and make efforts to increase general awareness of and appreciation for mathematics. This course is one attempt to shed light on some of the important "roots of the subject," and will thus be somewhat different from the usual mathematics course in that, while the details of the mathematics itself will certainly not be neglected, there will be two other important components of the course. First is an historical/biographical emphasis. Mathematics has been, and continues to be, a major cultural force in civilization, and mathematicians necessarily work within the context of their time and place in history. Because only a few episodes and personalities can be highlighted in a course like this, it should be kept in mind that major results in mathematics come about not as isolated flashes of brilliance, but after years (or even centuries) of intellectual struggle and development. Second is an attempt to provide some insight into the nature of mathematics and those who create it. Mathematics is a living, dynamic and vast discipline. The American Mathematical Society's 1979 subject classification contains 61 basic classifications having approximately 3400 subcategories, and it has been estimated that the number of new theorems published yearly in mathematics journals is in excess of 100,000. It is hoped that this course will give the student some perspective on mathematics as a whole, and also provide some insight on how mathematics has developed over the years. The principal objective of the course, however, is that the student gain an understanding of the mathematics itself. In these notes, statements and proofs of theorems are often given in a form as close as possible to the original work. In practically all fields of scholarship, a valuable piece of advice is to read the classics, but this is not heard as much in mathematics as in some other areas. For one thing, the mathematics of previous generations is often difficult to read and not up to modern standards of rigor. Nevertheless, by reading primary sources, much valuable insight into a subject can be gained, and this is true for mathematics as well as other fields. The effort required is justified by the benefits.

Deciding which theorems to include has been a major part of the preparation of this course, and the final list is bound to reflect personal taste. Among the guidelines used to make the choices were accessibility to students with a calculus background, variety in the branches of mathematics represented, inclusion of the "superstars" in mathematics, and the intellectual quality of the results. After some thought, it was decided that being able to cover the complete proof of a theorem was important but not necessary. Thus, a few topics, such as Godel's theorem, are included with considerable discussion, but whose proofs require more mathematics than can reasonably be expected from the students at this stage in their mathematical development.

Finally, a few words to the students about what is expected of them with respect to the proofs of the theorems. Memorization of the proofs should not be a primary goal, although you may find that a certain amount of memorization occurs as a by-product as you work toward the main goal, which is understanding. An effective way to study mathematics is to read the material three times (at least); the first time read only the definitions and the statements of the theorems to get an idea of the mathematical setting; next re-read the material, this time scanning the proofs, but not checking all the details, in order to see what general techniques are used; finally, with pencil and paper ready at hand, read everything carefully, making sure you fully understand the logical path chosen by the mathematician to construct the proof. In many of the proofs in these notes details or steps have been omitted in some places. Whenever this occurs, the student is expected to supply the missing parts. Often, but not always, a statement such as "details are left to the student" helps identify gaps in the arguments. Understanding the great theorems of mathematics will certainly require effort, concentration and discipline on the part of the student; after all, these theorems do represent pinnacles of mathematical thought. Those who persevere and gain an understanding of these theorems will, however, also gain a sense of personal satisfaction that comes from being able to comprehend some of the masterpieces of mathematics.