A DOZEN MINIMA FOR A PARABOLA

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On the parabola defined by $y = x^2$, let $P = (a, a^2)$ be any point except the vertex. Because of symmetry, we can assume that $a > 0$. The normal line to the parabola at P will intersect the parabola again at, say, Q. The region bounded by the parabola and its normal line will be called the *parabolic segment*. See Figure 1.

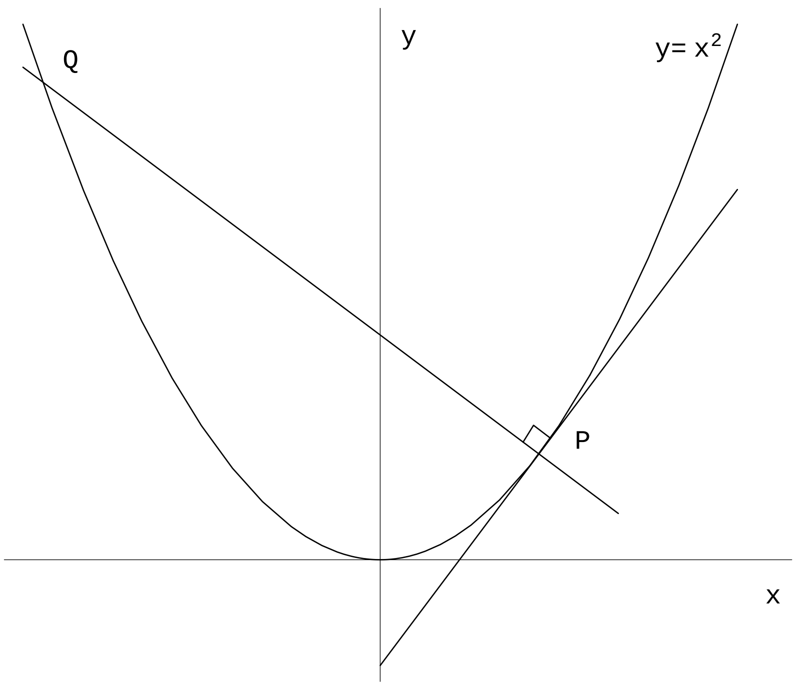

Figure 1

This setting is a rich source of minimization problems involving several calculus concepts. Among these problems there are at least twelve different values of a that minimize some some interesting quantity such as the distance between P and Q. All the problems can be set up by hand, and some can be finished by hand, but for others a computer algebra system is eventually a welcome aid. In several of the problems, the function to be minimized is defined by a definite integral, with the independent variable a in one or both limits of integration, and usually in the integrand as well. In such cases, the calculations are sometimes simpler using the Fundamental Theorem of Calculus or its extension, Leibniz' Rule, as opposed to evaluating the integrals first.

The Missouri Section of the MAA has used one of these problems as the lead question on the Missouri MAA Collegiate Mathematics Competition, a team problem solving event held annually in conjunction with the section meeting since 1996. Information about the competition and lists of problems and solutions can be found on the Missouri Section webpage, http://momaa.math.umr.edu/. The author first heard version 2b of the problem from the late Bob Krueger when we were colleagues at the University of Nebraska.

Here are the problems. Those sharing the same number have the same solution.

For the parabola $y = x^2$ and $a > 0$ consider the normal line to the parabola at $P = (a, a^2)$, which intersects the parabola again at Q. Find the value of a which minimizes:

1a. *The y-coordinate of Q.*

1b. *The length of line segment PQ.*

2a. *The horizontal distance between P and Q.*

2b. *The area of the parabolic segment. (The area turns out to be one-sixth of the cube of the horizontal distance in 3a.)*

2c. *The volume of the solid formed by revolving the parabolic segment around the vertical line k units to the right of P or k units to the left of Q, where $k \geq 0$. (Refer to problem 3 and try using Pappus' Theorem instead of the usual integral.)*

3. *The y-coordinate of the centroid of the parabolic segment. (The x-coordinate of the centroid is always $x = -\frac{1}{4a}$.)*

4. *The length of the arc of the parabola between P and Q.*

5. *The y-coordinate of the midpoint of the line segment PQ.*

6. *The area of the trapezoid bounded by the normal line, the x-axis, and the vertical lines through P and Q.*

7. *The area bounded by the parabola, the x-axis, and the vertical lines through P and Q.*

8. *The area of the surface formed by revolving the arc of the parabola between P and Q around the vertical line through P.*

9. *The height of the parabolic segment (i.e., the distance between the normal line and the tangent line to the parabola that is parallel to the normal line).*

10. *The volume of the solid formed by revolving the parabolic segment around the x-axis.*

11. *The area of the triangle bounded by the normal line, the vertical line through Q, and the x-axis.*

12. *The area of the quadrilateral bounded by the normal line, the tangent line, the vertical line throught Q, and the x-axis.*

Exact answers can be found, sometimes with the aid of the computer, for all except problem 8, but some of the exact answers come from solving cubic or quartic equations and are not simple. For instance, in one of the problems, the minimum occurs when

$$a = \frac{1}{12}\sqrt{\frac{1}{2}(-24 + (276480 - 69120\sqrt{11})^{1/3} + 24(5(4+\sqrt{11}))^{1/3})},$$

which has the much more useful decimal form $a = 0.564641\ldots$.

Here are the twelve distinct solutions (values of a) in increasing order. Readers and their students are invited to match the solutions with the problems.

$$\left\{\frac{1}{2\sqrt{2}},\ \frac{1}{2},\ \sqrt{\frac{3}{10}},\ 0.558480,\ 0.564641,\ 0.569723,\ 0.574646,\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt[4]{8}},\ \frac{1}{\sqrt[4]{6}},\ 0.644004,\ \frac{1}{\sqrt{2}}\right\}$$