Read the directions carefully.
Each question is worth 20 points,
with a maximum of 100 points possible.
Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.
1. Consider the differential equation \( \frac{dy}{dx} = (y - 2)^2(y^2 - 9) \).

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order nonlinear - coefficient depends on \( y \).
autonomous - \( x \) doesn't show up in the equation
separable - can be written as \( \frac{dy}{dx} = f(x,y) = g(x)h(y) \) where \( g(x) = 1 \).

B. Find the critical points for the differential equation. Classify each point as either asymptotically stable, unstable, or semi-stable. Draw the appropriate phase portrait.

CP: \( y = -3, 2, 3 \)

\[
\begin{array}{c|c|c|c}
\text{Int} & TV & +/- & \uparrow/\downarrow \\
\hline
(-\infty, -3) & -4 & + & \uparrow \\
(-3, 2) & -1 & - & \downarrow \\
(2, 3) & 2.5 & - & \downarrow \\
(3, \infty) & 4 & + & \uparrow \\
\end{array}
\]

- \( y = 3 \) unstable
- \( y = 0 \) semi-stable
- \( y = -3 \) asymptotically stable
2. Solve the initial value problem \( \frac{dy}{dx} = x(y - y^2) \); \( y(0) = \frac{1}{2} \) (an implicit solution is acceptable)

\[
\frac{dy}{dx} = xy(1-y)
\]

1st order non-linear, autonomous, separable

\[
\Rightarrow \int \frac{dy}{y(1-y)} = \int x \, dx
\]

\[
\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{1}{y} + \frac{1}{1-y}
\]

\[
\Rightarrow 1 = A(1-y) + By
\]

\[y = 0 \Rightarrow 1 = A\]

\[y = 1 \Rightarrow 1 = B\]

\[
\Rightarrow \int \left( \frac{1}{y} + \frac{1}{1-y} \right) dy = \int x \, dx
\]

\[u = 1-y\]

\[\text{du} = -dy\]

\[
\ln|y| - \ln|1-y| = \frac{1}{2} x^2 + C
\]

\[
\ln \left| \frac{y}{1-y} \right| = \frac{1}{2} x^2 + C
\]

\[y(0) = \frac{1}{2} \Rightarrow \ln \left| \frac{\frac{1}{2}}{1-\frac{1}{2}} \right| = 0 + C \Rightarrow C = 0
\]

\[
\ln \left| \frac{y}{1-y} \right| = \frac{1}{2} x^2 \text{ implicit solution}
\]

Or

\[
\frac{y}{1-y} = e^{\frac{1}{2} x^2} \Rightarrow y = (1-y) e^{\frac{1}{2} x^2} = e^{\frac{1}{2} x^2} - e^{\frac{1}{2} x^2} y
\]

\[
\Rightarrow (1+e^{\frac{1}{2} x^2}) y = e^{\frac{1}{2} x^2}
\]

\[
\Rightarrow y(x) = \frac{e^{\frac{1}{2} x^2}}{1+e^{\frac{1}{2} x^2}} \text{ explicit solution}
\]
3. Consider the differential equation \( xy' + (x + 1)y = e^{-x} \sin(2x) \).

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order
- linear - equation of the form \( a_1(x)y' + a_0(x)y = g(x) \)
- nonautonomous - \( x \) is in the equation
- nonseparable - cannot be rewritten as \( \frac{dy}{dx} = g(x) h(y) \)

B. Give a maximal interval \( I \) over which the solution is defined.

Std form: \( y' + \left( \frac{x+1}{x} \right)y = e^{-x} \sin(2x) \)

\( x > 0 \) or \( x < 0 \)

C. Find an explicit solution to the differential equation.

\[
\begin{align*}
IF &= \int P(x) \, dx = e^\int \left(1+\frac{1}{x}\right) \, dx = e^{x + \ln x} = x e^x, \quad x > 0 \\
xe^x \left[ y' + \left( \frac{x+1}{x} \right)y \right] &= xe^x \left( \frac{e^{-x} \sin(2x)}{x} \right) = \sin(2x) \\
x e^x y' + (x+1)e^x y &= \sin(2x) \\
\frac{d}{dx} \left[ xe^x y \right] &= \sin(2x) \\
\int \frac{d}{dx} \left[ xe^x y \right] \, dx &= \int \sin(2x) \, dx \\
x e^x y &= -\frac{1}{2} \cos(2x) + C \\
y(x) &= \frac{-\cos(x)}{2x e^x} + \frac{C}{x e^x}
\end{align*}
\]
4. Newton's law of cooling states that the rate of change in the temperature of a body is proportional to the difference in the temperature of that body and temperature of the surrounding medium. Now suppose that a murder victim is discovered at midnight with a recorded temperature of 31°C. An hour later, the temperature of the victim is 29°C. Assume that the temperature of the surrounding air remains a constant 21°C. Calculate the victim's time of death. Note: the "normal" temperature of a living person is 37°C.

\[
\frac{dT}{dt} = K(T - 21)
\]

\[
\begin{align*}
T(0) &= 31 \\
T(1) &= 29
\end{align*}
\]

\[
\Rightarrow \int \frac{dT}{T - 21} = K \int dt
\]

\[
\ln|T - 21| = Kt + C
\]

\[
T(t) - 21 = e^{Kt+C}
\]

\[
T(t) = 21 + C_1e^{Kt}
\]

\[
T(0) = 31 = 21 + C_1e^0
\]

\[
\Rightarrow C_1 = 10
\]

\[
\Rightarrow T(t) = 21 + 10e^{Kt}
\]

\[
T(1) = 29 = 21 + 10e^K
\]

\[
\Rightarrow e^K = \frac{4}{5}
\]

\[
\Rightarrow K = \ln\left(\frac{4}{5}\right)
\]

\[
\Rightarrow T(t) = 21 + 10e^{\ln\left(\frac{4}{5}\right)t}
\]

Now let \(T(t_c) = 37 = 21 + 10e^{\ln\left(\frac{4}{5}\right)t_c}\)

\[
\Rightarrow \ln\left(\frac{4}{5}\right)t_c = \ln\left(\frac{8}{5}\right)
\]

\[
\Rightarrow t_c = \frac{\ln\left(\frac{8}{5}\right)}{\ln\left(\frac{4}{5}\right)} \approx -2.106 \approx -2\text{hrs 6mins}
\]

\[
\Rightarrow \text{Time of death: 9:54 PM.}
\]
5. Use reduction of order to find a second linearly independent solution to the differential equation \( xy'' + (1 - 2x)y' + (x - 1)y = 0, \quad x > 0 \), where \( y_1(x) = e^x \) is a known solution.

(NO POINTS for the integral formula).

1. Assume \( y_2(x) = u(x)y_1(x) = u(x)e^x \) is a solution
   \[
   y_2' = u'e^x + u e^x \\
   y_2'' = u''e^x + 2u'e^x + u e^x
   \]

2. Plug in \( y_2 \)
   \[
e^x \left[ x(u'' + 2u' + u) + (1 - 2xu' + u) + (x - 1)u \right] = 0
   \]

3. Regroup by \( u \)
   \[
xu'' + (2x + 1 - 2x)u' + (x + 1 - 2x + x - 1)u = 0 \\
xu'' + u'' = 0
   \]

4. Change of Variables
   \[
   \begin{align*}
   \text{Let} \quad & w = u' \quad \Rightarrow \quad x \frac{dw}{dx} + w = 0 \\
   \rightarrow & \quad \frac{w'}{w} = -\frac{1}{x} \\
   \Rightarrow & \quad w = K_1 x^{-1} = u'
   \end{align*}
   \]

5. Solve for \( w \)
   \[
   \int \frac{dw}{w} = -\int \frac{dx}{x} \quad \Rightarrow \quad \ln|w| = -\ln|x| + K \\
   \Rightarrow & \quad w = K_1 x^{-1} = u'
   \]

6. Solve for \( u \)
   \[
   u = K_1 \int \frac{dx}{x} = K_1 \ln x + K_2 \\
   \quad \text{Pick} \quad \begin{cases} K_1 = 1 \\
   K_2 = 0 \end{cases}
   \]

7. Find \( y_2(x) = u(x)y_1(x) \)
   \[
   = e^x \ln x
   \]
**Bonus (10 points):** Consider the differential equation \( \frac{dy}{dx} = \sqrt{y - x^2 + 1} \). Determine a region \( R \) in the \( xy \)-plane for which the differential equation would have a unique solution through each point \((x_0, y_0)\). Sketch the region.

\[
\frac{dy}{dx} = f(x, y) = \sqrt{y - x^2 + 1}
\]

\[
\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y - x^2 + 1}}
\]

\[
R = \{ (x, y) : y - x^2 + 1 > 0 \} \cup \{ (x, y) : y > x^2 - 1 \}
\]