Review of Integration Techniques

A. Integration by Parts: If u and v are functions of x and have continuous derivatives, then
\[ \int u dv = uv - \int v du. \]

Guidelines for picking u:
- Inverse trigonometric functions (arctan(x), arccsc(x), etc.)
- Logarithms (ln(x))
- Algebraic functions (polynomials: 3x^2, x^{10}-x, etc.)
- Trigonometric functions (sin(x), cos(2x), tan(2x), etc.)
- Exponential functions (e^x, 2^x, etc.).

Ex: \[ \int x \sin(x) \, dx = -x \cos(x) - \int (-\cos(x)) \, dx \]
\[ \quad = -x \cos(x) + \int \cos(x) \, dx \]
\[ \quad = -x \cos(x) + \sin(x) + C \]

B. Partial Fractions

Decomposition of \( \frac{N(x)}{D(x)} \) into Partial Fractions

1. Divide if improper: If degree of \( N(x) \) ≥ degree of \( D(x) \), then divide to get
   \[ \frac{N(x)}{D(x)} = \left( a \text{ polynomial} \right) + \frac{N_1(x)}{D(x)} \]
   where degree of \( N_1(x) \)
   \[ \quad < \text{degree of } D(x) \]
2. Factor \( D(x) \): Factor the denominator into factors of the form
   - \( (px+q)^m \) (linear)
   - \( (ax^2+bx+c)^n \) (quadratic) \[ \text{where } ax^2+bx+c \text{ is irreducible} \]
3. **Linear factors**: For each factor of the form \((px+q)^n\), the partial fraction decomposition must include the following sum of \(m\) fractions:

\[
\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \ldots + \frac{A_m}{(px+q)^n}
\]

4. **Quadratic factors**: For each factor of the form \((ax^2+bx+c)^n\), the partial fraction decomposition must include the following sum of \(n\) fractions:

\[
\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \ldots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}
\]

**Example**

\[
\int \frac{5x^2+20x+6}{x^3+2x^2+x} \, dx
\]

\[
5x^2+20x+6 = 5x^2+20x+6
\]

\[
x^3+2x^2+x = x(x+1)^2
\]

\[
= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}
\]

Now multiply both sides by the least common denominator to get

\[
5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx+1
\]

To solve for \(A\), let \(x=0\)

\[
0 = A(1) + 0 + 0 \implies A = 6
\]

To solve for \(C\), let \(x=-1\)

\[
5-20+6 = -9 = 0 + 0 - C \implies C = 9
\]
To solve for $B$, let $x = 1$

$$5 + 20 + 6 = A(4) + B(2) + C$$

$$31 = 6(4) + 2B + 9$$

$$-2 = 2B \quad \Rightarrow \quad B = -1$$

Now,

$$\int \frac{5x^2 + 20x + 6}{x(x+1)^2} \, dx = \int \left( \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2} \right) \, dx$$

$$= 6 \int \frac{dx}{x} - \int \frac{dx}{x+1} + 9 \int \frac{dx}{(x+1)^2}$$

$$= 6 \ln|x| - \ln|x+1| + \frac{9}{x+1} + C$$

$$= \ln|x^6| - \ln|x+1| - \frac{9}{x+1} + C$$

$$= \ln \left| \frac{x^6}{x+1} \right| - \frac{9}{x+1} + C$$