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Reconstructing 3-D Block Size Distributions from 2-D Measurements on Sections

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ABSTRACT: Measurement of particles, whether grains on a microscope slide or image of a muck or stock pile, is two dimensional. Principles of geometric probability and stereology can be used to reconstruct or unfold a three dimensional size distribution. This analytical solution or unfolding function can be calibrated with an empirical calibration function.

1 INTRODUCTION

The problem of determining the true block size distribution of blast fragmentation is one of measuring, on the surface of an assemblage of blocks, some two dimensional size parameter of the individual blocks and transforming it into a three dimensional block size distribution.

Similar problems exists in the fields of biology, metallography, and petrography, i.e. to obtain true particle size distributions of grains or bodies embedded in a three dimensional volume from measurements on a two dimensional section or cut. For this type of problem, closed form solutions based on geometric probabilities exist, and are part of a discipline known as stereology (DeHoff and Rhines, 1968; Underwood, 1970; Weibel, 1980; 1981; Russ, 1986).

2 PRINCIPALS OF GEOMETRIC PROBABILITY

2.1 Geometric Probability

Geometric probabilities are probabilities of geometric occurrences. An example of such would be to calculate the probability of a certain number of particles intersecting a random sampling plane within a given volume (DeHoff and Rhines, 1968).

2.2 Stereology

Stereology deals with a body of methods for the exploration of three-dimensional space when only two-dimensional sections through solid bodies or their projections are available (Elias, 1967). The aspects of stereology that are pertinent in the context of measuring fragmentation are those dealing with he size distributions of particles derived from the size distribution of their sections (Santalo, 1976).

In the notation of Santalo, the true size distribution of a body of particles is expressed as H(l) and the observed profile distribution on a planar cut or linear transect through that body is expressed as h(s), where l is some measure of the true particle size and s is a similar measure of observed particle profile size. The task of a stereological solution is to infer H(l) given h(s). This process in known in stereology as "unfolding" a distribution.

Unfolding is particularly difficult because the observed profile size of a particle is a function of both the true size of the particle, and of the nature of the intersection between the particle and the sampling line or plane. Because it is impossible to determine whether a small profile is derived from a small particle sampled through its largest dimension, or from intersecting a small corner of a larger particle, it necessary to use geometric probabilities and make a-priori assumptions about H(l) to reconstruct it.

2.3 Assumptions and Applicability of Stereology

Stereological particle size relationships are generally applied to polydispersed systems of particles.

In general, the following assumptions are made:

1. The particles are randomly distributed in the volume.
2. If the particles have an anisotropic shape, (any shape other than spherical) they are randomly oriented.
3. The particles are convex.
4. The sampling plane (section) or sampling line is applied at a random orientation.

In addition, most closed form solutions make a-priori assumptions about the following:

1. The form of the size distribution of particles.
2. The shape of the particles.

Particle shapes are practically limited to simple spheres, cubes, spheroids, etc. A simplification is often made in which particles are modeled by the closest matching regular geometric shape. By far the most solutions are for spheres.

In the case of particles which are irregular, such as lumps resembling spheres, a measurement such as a diameter depends on which part of the particle is measured. In this case an "equivalent" measure can be used. An example is an area equivalent diameter d_{ea}, defined as the diameter of a circle of area equal to a measured cross sectional area, where:

$$d_{ea}^2 = \frac{4}{p} a \qquad [1]$$

and a is the area of a non-circular profile.

2.4 Basic Stereological Relationships

From Santalo (1976), if a volume of randomly oriented and randomly distributed particles, where the number of particles per unit volume is N_V, is intersected by a sectioning plane, then the number of particles per unit area (N_A) on the sectioning plane is given by:

$$N_A = \frac{M}{2p} N_V \qquad [2]$$

where M is the mean curvature of the particles. For spheres, M=2● d, where d is the mean diameter of the spheres. The equation becomes:

$$N_A = dN_v \qquad [3]$$

For cubes, M=3● a, where a is the mean edge length of the cubes. The equation becomes:

$$N_A = \frac{3}{2} a N_V \qquad [4]$$

If the volume is intersected by a linear traverse, then the number of particles per unit length (N_L) on the line section is given by:

$$N_L = \frac{f}{4} N_V \qquad [5]$$

where f is the surface area of the particles. For spheres, f=4● r^2, where r is the diameter of the spheres. The equation becomes:

$$N_L = p\, r^2 N_V \qquad [6]$$

For cubes, $f=6a^2$, where a is the edge length of the cubes. The equation becomes:

$$N_L = \frac{3}{2} a^2 N_V \qquad [7]$$

2.5 Some Basic Stereological Solutions

The stereological solutions given in the literature in general try to reconstruct the true size distribution H(l) from the measured profile size distribution h(s). They do so by assuming a particle shape (usually spherical), a distribution (usually normal), and try to solve for example, $N_A = f(N_V)$ for discrete diameter or radius classes.

The number of classes (m) and the class widths (✎) are the same for both N_A and N_V, however the classes are labeled j=1,m for N_V and i=1,m for N_A using the notation of Weibel (1980). From geometric probabilities, it is known that a contribution to $N_V(j)$ can be made from $N_A(i)$, $N_A(i-1)$, ..., $N_A(1)$ where i=j.

The method of Saltykov (Weibel, 1980) solves a diagonal matrix k_{ij} representing the relative probabilities from the formulation:

$$N_A(i) = d \sum_{j=i}^{m} k_{ij} N_V(j) \qquad [8]$$

If the formulation is reversed, N_V can be determined from N_A:

$$N_V(j) = \frac{1}{d} \sum_{i=j}^{m} a_{ij} N_A(i) \qquad [9]$$

where a_{ij} is the inverse of k_{ij}.

Similar solutions abound in the literature. The method of Bach (1965) is identical, but allows for sampling planes of finite thickness.

A second method of Saltikov (1967) allows for particles of any shape, provided they are all of that shape, and assumes a log normal particle size distribution. The methods of Spektor (1950), Lord and Willis (1951), and Cahn and Fullman (1956) use linear samples rather than plane areas, where chord lengths N_L are measured.

3 THE APPLICABILITY OF SECTIONING METHODS TO FRAGMENTATION

In applying any of the above methods to the problem of reconstructing a block size distribution of a pile of blast fragmented rock from a measurement made on the surface of the pile, many of the underlying assumptions of these methods are violated.

One of the violated assumptions is that of a randomly oriented infinitely thin sampling plane. Rather than a random plane, the surface of the rock pile will be sampled. This sampling "plane" will be neither infinitely thin as required by most stereological solutions, nor of constant thickness as accommodated by at least one solution. The effective thickness of the sampling surface will vary, as gaps between particles on the surface of the pile will reveal particles one or two layers deep into the pile. The profiles that will be measured are not the intersection of the particles with a sampling plane, but rather a "projected" profile, where the largest dimension of the fragment, in the direction of projection, is revealed. Furthermore, particles in the second or third layer into the pile will be partially obscured or overlapped by particles in the first layer.

Other assumptions that may be violated are:

1. The particles will most likely not be of a simple geometric shape, perhaps not even convex.

2. The size distribution of the particles will be largely unknown.

The assumptions of random distribution and orientation of particles may not be valid.

Two stereological solutions (Bach, 1965; Saltikov, 1967) were tried out on crushed rock assemblages of known size distribution. In general, the results were unsatisfactory relying too heavily on a-priori knowledge of actual parameters of the distribution (Maerz, 1990),. In response to this a new method of unfolding or reconstructing fragment size distributions was developed.

4 A NEW METHOD OF UNFOLDING DISTRIBUTIONS

For the purposes of analyzing fragmentation using image analysis, a quick and automated unfolding solution was required. The one that was finally adapted was based on equation [3], assuming that blocks are spherical.

A further assumption was made that if the observed distribution h(s) is divided into a number of classes of equal class width, equation [3]. could be applied to each class:

$$N_V(d) = \frac{1}{d} N_A(d) \qquad [10]$$

where d is the mean diameter of each class.

To address the difference between sampling the surface of an assemblage of particles and sampling an infinitely thin plane through the assemblage (as assumed for equation 2), a simple experiment was conducted.

An assemblage of several hundred 24 mm styrofoam balls was placed in a box, carefully avoiding regular packing. The surface of this assemblage was photographed. The observed (measured) distribution of diameters (d_{ea}) in Figure 1 differed from the distribution of diameters expected from sampling an infinitely thin plane.

The observed distribution was bi-modal, where one mode represented non overlapped balls exposed at the surface of the assemblage, and the second mode represented balls which were partially overlapped in the second and third layer of the assemblage. Although these results are not entirely relevant to multi-size and shape assemblages, the experiment showed that partially overlapped blocks not in the top layer of the pile can be measured.

4.1 Calibration of the solution

As a result of the above experiment, a calibration function f(d) was added to the unfolding equation (Equation 10):

$$N_V(d) = \frac{1}{d f(d)} N_A(d) \qquad [11]$$

For each diameter class (d) there is a different calibration factor f(d). The set of calibration factors for all the classes is known as the calibration function.

With the addition of the calibration function, Equation 11 becomes semi-empirical, because f is an empirically derived calibration function. The significance of f is that it accounts for any systematic differences between the theoretical solution for a polydispersed system of spheres (Equation 10), and the actual solution for fragmentation. The physical significance of f is to account for a combination of the following factors:

1. The effect of overlap of fragments. The effect of different profile sizes because of varying degrees of overlap is similar to but not identical to the stereological situation where a sphere is sectioned at various places.

2. The effect of missing fines. A proportion of the smaller sizes is typically missing in photographs of fragmentation by virtue of falling between and behind larger fragments, and because of insufficient photographic resolution. This problem has been cited in typical stereological applications as well (DeHoff and Rhines, 1968).

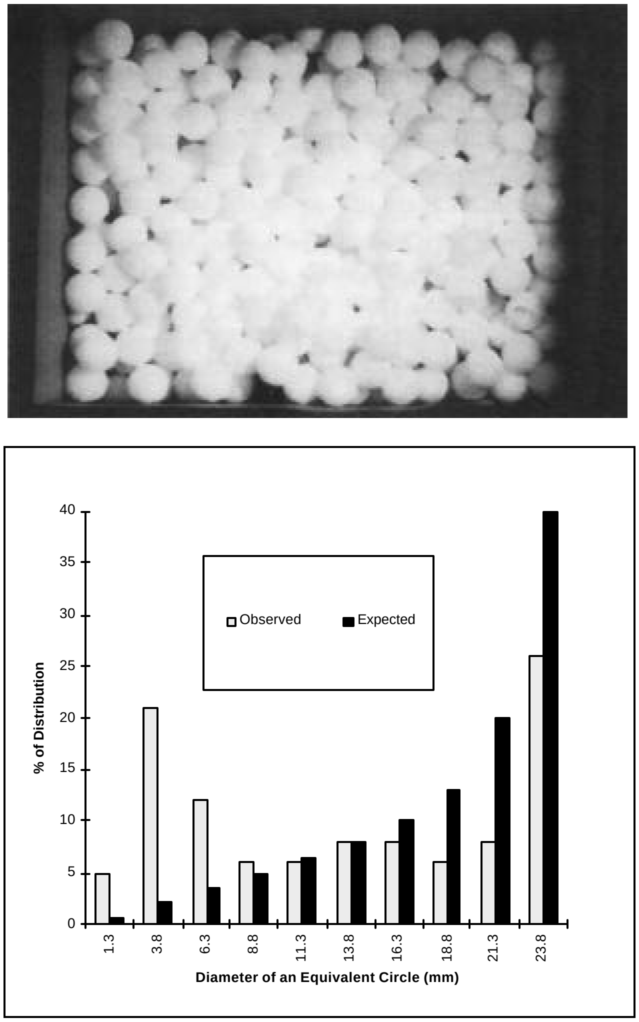

Figure 1. Top: Assemblage of 24 mm spheres; Bottom: Expected distribution of diameters (circular area equivalent) from a true section, based on Monte Carlo simulation of geometric probabilities, and observed distribution of diameters (circular area equivalent), of the assemblage of spheres using image analysis.

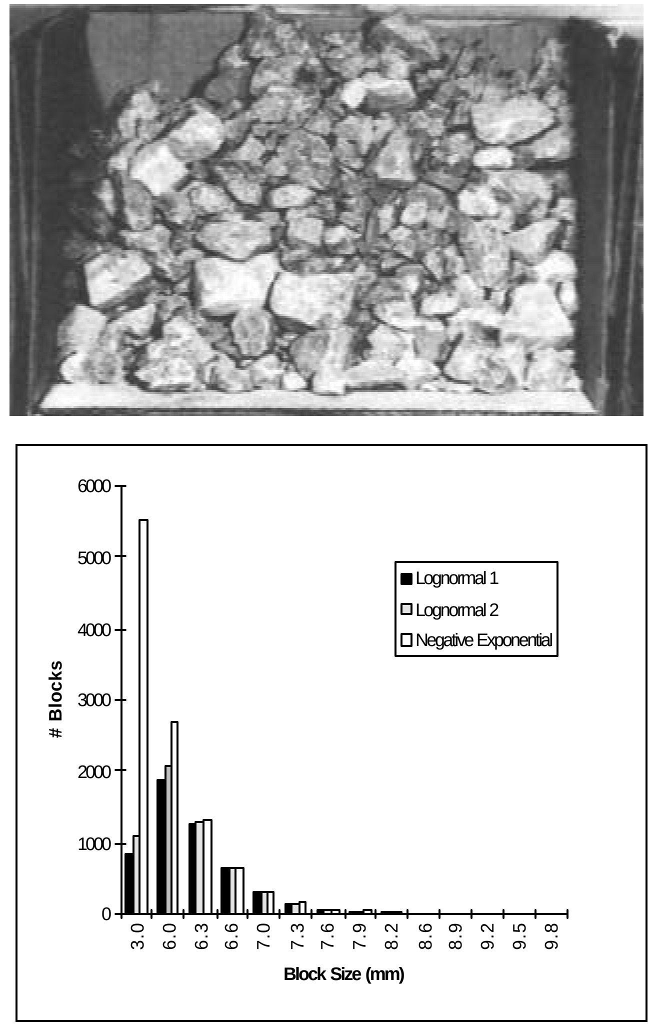

Figure 2. Top: Assemblage of crushed rock, dumped and photographed in a box. Bottom: Distributions of crushed rock: 1) lognormal, mean of 8.75 mm and standard deviation 9.92 mm; 2) lognormal, mean of 8.40 mm and standard deviation of 9.59 mm; 3) negative exponential distribution with an arithmetic mean of 6.19 mm.

3. The effect of the shape of the distribution. Although there is no theoretical a-priori assumption of distribution shape inherent in the method, different distributions have varying amount of fines, and consequently varying amounts of missing fines.

The calibration function can be determined by back calculation for any known size distribution. Three populations of crushed rock samples were made up by sieving and mixing, one negative exponential distribution, and two lognormal distributions (Figure 2). The rock was dumped into a box to simulate loading in the back of a haulage truck, photographed and processed using image analysis. The calibration functions were back calculated using equation 11, for each of the distributions, and are shown in Figure 3.

For the most part the value of the calibration function is close to 1, i.e. it does not affect the outcome significantly. The value deviates from 1 at the coarse and fine ends, for two different reasons.

At the coarse end (class #10) the value is affected by sampling error, since there are very coarse blocks, e.g. the presence of a single coarse block could be over representative, while the absence of the same block could be under representative. At the fine end (class #1) the problem is simply missing fines, as the undersize blocks tend to fall between and behind the large blocks, or are not resolved by the image analysis system.

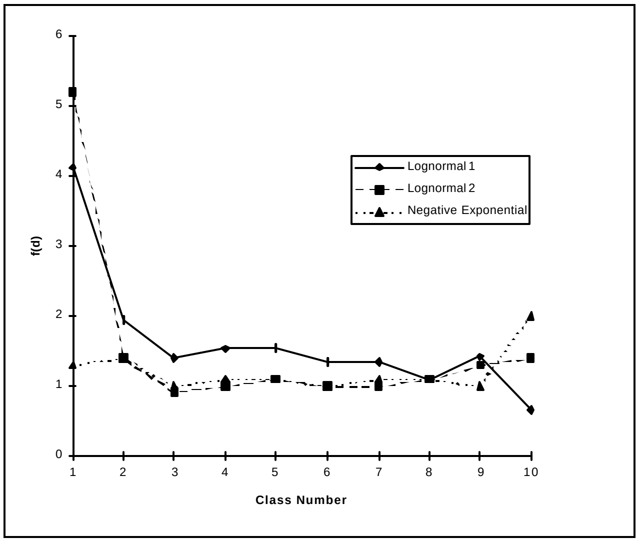

Figure 3. Calibration function derived for all three distributions. For the most part f has a value close to 1, subject to random sampling errors.

5 CONCLUSIONS

Principles of geometric probability and stereology can be used to reconstruct or unfold a three dimensional size distribution, from two dimensional measurements on the surface of a rock assemblage. This analytical solution or unfolding function can be calibrated with an empirical calibration function.

REFERENCES

Bach, G. 1965. Kugelgrößenverteilung und Verteilung der Schnittkreise; ihre wechselseitigen Beziehungen und Verfahren zur Bestimmung der einen aus der anderen. Proc. Symp. on Quantitative Methods in Morphology, pp. 23-45.

Cahn, J. W., and Fullman, R. L. 1956. On the use of lineal analysis for obtaining particle size distribution functions in opaque samples. Trans. AIME 197, pp. 610.

DeHoff, R. T., and Rhines, F. N. 1968. Quantitative microscopy. New York, McGraw-Hill, 422 pp.

Elias, H. (ed.) 1967. Stereology. Proc. 2nd Int. Cong. Stereology.

Lord, G. W., and Willis, T. F. 1951. Calculation of air bubble size distribution from results of a rosiwal traverse of aerated concrete. ASTM Bull. 177, pp. 56.

Maerz, N. H. 1990. Photoanalysis of Rock Fabric. Ph. D. Thesis, Dept. of Earth Sciences, University of Waterloo, 227 pp.

Russ, J. C. 1986. Practical Stereology. Plenum Press, New York, 185 pp.

Saltikov, S. A. 1967. The determination of the size distribution of particles in an opaque material from a measurement of the size distribution of their sections. Proc. 2nd. Int. Cong. for Stereology, pp. 163-173.

Santalo, L. A. 1976. Integral geometry and geometric probability. London, Addison-Westly, 404 pp.

Spektor, A. G. 1950. Analysis of distribution of spherical particles in non-transparent structures. Zavod. Lab. 16, pp. 173.

Underwood, E. E. 1970. Quantitative stereology. Reading Massachusetts, Addison-Wesley, 274 pp.

Weibel, E. R. 1980. Stereological methods. Vol. 1. Practical methods for biological morphometry. London, Academic Press, 415 pp.

Weibel, E. R. 1981. Stereological methods. Vol. 2. Theoretical foundations. London, Academic Press, 339 pp.