37. Construct the wave functions $\phi_n(q)$ and $\phi_n(x)$ for the lowest four harmonic oscillator eigenstates. (Read the section in the text about how to go from the dimensionless version of the wave function $\phi_n(q) = \langle q | n \rangle$, to the dimensioned form $\phi_n(x) = \langle x | n \rangle$.)
38. Up till this point we have considered particles with zero spin. Now consider a particle of spin $\frac{1}{2}$ (such as an electron). Assume that the state space of such a particle is spanned by a complete ONB of states $\{|\vec{r}', s_z\rangle\}$, each of which corresponds to a particle at a specific point $\vec{r}' \in \mathbb{R}^3$ and in a particular “spin state”, with the $z$-component of spin $s_z$ taking on just two possible values, $s_z \in \{\frac{1}{2}, -\frac{1}{2}\}$. The state with $s_z = \frac{1}{2}$ is said to be “spin-up”, that with $s_z = -\frac{1}{2}$ is said to be “spin-down” along the $z$-axis. Let $S_z$ be the operator such that $S_z |\vec{r}', s_z\rangle = s_z |\vec{r}', s_z\rangle$.

(a) Show that (or explain why) these two operators commute, i.e., $[\vec{R}, S_z] = 0$.

(b) Write down completeness and orthonormality relations for the ONB $\{|\vec{r}', s_z\rangle\}$. Note that these states have both a continuous index and a discrete one, so that one has to do the correct kind of summation, and use the correct delta function for each index.

(c) Express an arbitrary state vector $|\psi\rangle$ for a spin $\frac{1}{2}$ particle as an expansion in this basis, showing that it requires two separate functions $\psi_+ (\vec{r})$ and $\psi_- (\vec{r})$, which need not be square-normalized and which you should express as inner products of the basis vectors with the state $|\psi\rangle$. In this situation one says that the system is described by a two-component wave function.

(d) Assuming that $\langle \psi | \psi \rangle = 1$, what joint normalization condition should these two functions satisfy for a physical state of the system? What is the physical interpretation of each function? If a measurement is made of the spin of the particle (with no information about its position being obtained in the process), how do we determine from our knowledge of these two functions the probability of measuring $s_z = 1/2$ or $-1/2$?
39. For a single particle whose position coordinate is \( \vec{r} \), the classical probability density \( \rho(\vec{r}_0) \) to find the particle at a point \( \vec{r}_0 \) is the delta function \( \rho(\vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \). (In other words a classical particle always has a well-defined position).

(a) Since this is just a function of the dynamical variable \( \vec{r} \), it corresponds in the quantum mechanical treatment to an operator \( \hat{\rho}(\vec{r}_0) = \delta(\vec{R} - \vec{r}_0) \). Obtain the ket-bra expansion of this operator in the position representation (in which it is diagonal), and show that its mean value taken with respect to a state \( |\psi\rangle \) is just \( \langle \hat{\rho}(\vec{r}_0) \rangle = |\psi(\vec{r}_0)|^2 \). (To compute the ket-bra expansion, you might start by computing the matrix elements of the operator \( \hat{\rho}(\vec{r}_0) \).)

(b) Classically, \( \vec{j} = \rho \vec{v} \) is the particle current density. In particular, the current density at \( \vec{r}_0 \) for a single particle is

\[
\vec{j}(\vec{r}_0) = \rho(\vec{r}_0) \vec{v} = \delta(\vec{r} - \vec{r}_0) \frac{\vec{P}}{m}.
\]

Construct a Hermitian operator \( \hat{J}(\vec{r}_0) \) that represents this classical observable, by promoting the classical variables to operators, and taking the Hermitian part of the resulting expression. Show that the mean value of this operator in the state \( |\psi\rangle \) is

\[
\langle \hat{J}(\vec{r}_0) \rangle = \frac{\hbar}{2im} [\psi^*(\vec{r}_0) \vec{\nabla} \psi(\vec{r}_0) - \psi(\vec{r}_0) \vec{\nabla} \psi^*(\vec{r}_0)].
\]

What can you conclude about the current density for a particle in a state for which the wave function is everywhere real? Determine the current density at \( \vec{r} \) for the Gaussian wave packet

\[
\psi(\vec{r}) = (2\pi a^2)^{-1/4} \exp \left( -\frac{r^2}{2a^2} \right) \exp \left( i\vec{E} \cdot \vec{r} \right).
\]
40. In the Schrödinger picture the state of the system is at each instant associated with a state vector $|\psi(t)\rangle$. This picture has the drawback that the state is defined only up to an overall phase, i.e., $|\psi\rangle$ and $|\psi'\rangle = e^{i\theta}|\psi\rangle$ both represent the same physical state. It is possible to remove this ambiguity by using another formulation of quantum mechanics in which the state of the system is represented at each instant not by a normalized vector $|\psi(t)\rangle$, but by an operator

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

which is usually referred to as the “density matrix”, and which is actually just the instantaneous projector $P_\psi$ on to the dynamical state $|\psi\rangle$ as it is defined in the Schrödinger picture. Show (or explain why) in the “density matrix” formulation of quantum mechanics, the density matrix $\rho(t)$ representing the quantum mechanical system of interest obeys the following properties:

(a) Show: At each instant of time the density matrix has unit trace, i.e., $\lim Tr[\rho(t)] = 1$.

(b) Show: The density matrix associated with $|\psi\rangle$ and $|\psi'\rangle = e^{i\theta}|\psi\rangle$ are associated with the same density matrix. Thus the density matrix avoids the ambiguity associated with the relative phase of the state vector.

**Solution:** By direct calculation one notes that

$$\rho = |\psi\rangle\langle\psi| \quad \rho' = e^{i\theta}|\psi\rangle\langle\psi|e^{-i\theta} = |\psi\rangle\langle\psi| = \rho.$$

(c) Show: The mean value of any observable $A$ at time $t$ is given by

$$\langle A(t) \rangle = \langle \psi(t) | A | \psi(t) \rangle = \lim Tr[\rho(t) A]$$

[Hint: choose an arbitrary discrete representation $\{|n\rangle\}$ in which to compute the trace.]

**Solution:** By direct calculation we find that

(d) Show: For a system evolving under the action of the Hamiltonian $H$, the density matrix obeys the so-called **Von Neumann equation** of motion

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)].$$

(e) Show: For a time independent Hamiltonian the density matrix at time $t$ can be obtained from the initial density matrix of the system through the relation:

$$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$$