9. A particle of mass \( m \) is confined to the interval \( x \in [0, \ell] \) by an infinite 1D potential well. [This is like an earlier problem, the results of which you should use here].

At a certain moment the particle is in a state associated with the wave function

\[
\psi(x) = \begin{cases} 
Ax(\ell - x) & \ell > x > 0 \\
0 & \text{otherwise}
\end{cases}
\]

(a) Sketch this wave function. What is the value of \( A \) that makes this wave function square normalized to unity?

(b) If a measurement of the particle’s position is made, what is the probability that it will be found in the interval \([0, \ell/4]\)?

(c) If, instead, an energy measurement is made when the particle is in this state, what values can be obtained? Express the probability to obtain each of those values in terms of an appropriate integral. Calculate numerically to three significant figures for the 3 lowest energies the probability that each could be obtained during an energy measurement performed on a particle in this state.
10. Let $|\psi\rangle = |x\rangle + \lambda |y\rangle$, where $\lambda$ is an arbitrary complex constant.

(a) Show only from the general rules given in the definition of the inner product $\langle \psi | \phi \rangle$ of two vectors, that with $|\psi\rangle$ as given above, it must be the case that $\langle \psi | = \langle x | + \lambda^* \langle y |$.

(b) Express the positive quantity $||\psi||^2 = \langle \psi | \psi \rangle$ in terms of $\lambda$, $|x\rangle$, and $|y\rangle$, using the basic rules associated with the inner product.

(c) By setting $\lambda = -(\langle y | x \rangle / \langle y | y \rangle)$, prove that

$$||x|| ||y|| \geq |\langle x | y \rangle|$$

which is known as Schwartz’s inequality. Under what conditions does the inequality become an equality?

(d) Use Schwartz’s inequality to show that if $\psi(x)$ and $\phi(x)$ are both square integrable functions, then so is $\chi(x) = \psi(x) + \phi(x)$, showing that $L^2(R)$ is closed under vector addition [Hint: it might help at some point to prove to yourself and use the fact that $|a + ib| > |b|$ for real $a, b.$]
11. Let $\psi(r)$ be a spherically symmetric wave function (independent of $\theta$ and $\phi$). Show that the 3-D transform/momentum-space-wavefunction

$$\hat{\psi}(k) = \frac{1}{(2\pi)^{3/2}} \int d^3r \ e^{-i\vec{k}\cdot\vec{r}} \psi(r)$$

of such a function can be reduced to a single radial integral as follows: (i) express the integral as a triple integral over spherical coordinates $(r, \theta, \phi)$, with volume element $d^3r = r^2 \sin \theta \ dr \ d\theta \ d\phi$. Since the integral is over all space, we can orient the coordinate system we integrate over in any direction that is convenient. So (ii) choose it so the $z$-axis lies along the direction of $\vec{k}$, so that $\vec{k} = k\hat{z}$, hence $\vec{k} \cdot \vec{r} = k z = kr \cos \theta$. (iii) Evaluate the $\phi$ integral. (iv) Now evaluate the integral over $\theta$, leaving a single integral over $r$. Note that the result is independent of the direction of $\vec{k}$, i.e., the Fourier transform of a spherically symmetric function is spherically symmetric in $k$-space.
12. A quantum mechanical particle is bound to a spherically symmetric force center in a state described by the wave function

$$\psi (\vec{r}) = A \frac{e^{-r/a}}{r}$$

where $a$ is a positive constant.

(a) Determine the value of the normalization constant (chosen real and positive) for which this state is square-normalized to unity.

(b) What is the probability that a position measurement will find the particle within a sphere of radius $a$ centered on the force center?

(c) Suppose the force center is suddenly annihilated by a cosmic ray so that the particle (still in this state but no longer bound), finds itself suddenly free. What is the probability $\rho(\hat{k})$ that a detector will find the particle to emerge with a momentum $\vec{p}_e = \hbar \hat{k}$? Is there any direction along which it is most likely to emerge?