21. Consider a triatomic molecule with three identical atoms that are bound together with each atom at its own corner of an equilateral triangle of edge length \( a \). An electron added to the molecule (to form a molecular ion) can be put in an identical atomic orbital on any one of the three atoms. Denote the atomic states in which the electron is on atom \( i \) as \( |i\rangle \), and assume that these three states \( \{|1\rangle, |2\rangle, \text{ and } |3\rangle\} \), form an orthonormal set. Set the zero of potential energy so that the mean energy associated with each such state vanishes, i.e., \( \langle i|H|i\rangle = 0 \) for each state \( |i\rangle \). Suppose also that the electron on an atom can move to either of its neighbors, such that

\[
\langle i|H|j\rangle = V_0 \quad \text{for } i \neq j
\]

(a) Construct a \( 3 \times 3 \) matrix \( [H] \) that represents the Hamiltonian within the subspace spanned by these 3 atomic states, using the states \( \{|i\rangle\} \) as an ONB for the subspace. Find the trace of this matrix.

(b) Find the energy eigenvalues and the degeneracies of this molecular ion.

(c) Construct an ONB of eigenstates \( \{\phi_n, \tau\} \) of the system, as linear combinations of the atomic states. (The \( \tau \) is included in the notation to allow for degenerate eigenvalues.)
22. Let \( \{|n\rangle\} \), with \( n = 0, 1, 2, \ldots \), be a discrete ONB for a quantum state space \( S \). Let \( A \) be an operator such that \( \langle n | A = \langle n + 1 | \), for all \( n = 0, 1, 2, \ldots \).

(a) Find matrix elements of \( A \) and \( A^+ \) in this representation. Show that \( A|0\rangle = 0 \).

(b) Write down ket-bra expansions of \( A \) and \( A^+ \) in this representation, explicitly showing the first few terms of the expansion. Be careful with summation limits. Is \( A \) Hermitian?

(c) Evaluate the action of \( A^+ A \) and \( AA^+ \) on \( |n\rangle \). Explicitly consider the case in which \( n = 0 \).

(d) Is \( A \) the inverse of \( A^+ \)? Is \( A \) unitary? Is \( A \) normal? (Recall: An operator \( A \) is normal if \( [A, A^+] = 0 \). Be careful.)
23. In a 2-dimensional space $S$ spanned by the orthonormal vectors $|1\rangle$ and $|2\rangle$, a certain linear operator $A$ has the following action:

\[ A|2\rangle = -i|1\rangle \quad A|1\rangle = i|2\rangle. \]

(a) Construct the matrix $[A]$ representing the operator $A$ in this representation. Is $A$ Hermitian? Is $A$ Unitary?

(b) Let $B$ be the linear operator that happens to be represented in this basis by a matrix $[B] = [A]^* \text{ that is the complex conjugate of that representing } A$. Construct $[B]$.

(c) Let $|\phi_1\rangle = i|1\rangle$ and $|\phi_2\rangle = |2\rangle$, be a new set of basis vectors. Show that these two new vectors are orthonormal.

(d) Construct the matrices $[A']$ and $[B']$ representing the operators $A$ and $B$ in this new basis, and show that $[B'] \neq [A']^*$.

What does this problem show about the complex conjugate of a linear operator?
24. Consider a so-called two level system, consisting of two states |1\rangle and |2\rangle separated in energy by an amount 2\Delta. Suppose the system is subject to a perturbation that “mixes” the two states, so that in the basis of states |1\rangle and |2\rangle the relevant Hamiltonian \( H \) is associated with a matrix

\[
[H] = \begin{pmatrix}
\epsilon_0 + \Delta & V \\
V & \epsilon_0 - \Delta
\end{pmatrix}.
\]

(a) Rewrite the matrix in the form \( \epsilon_0 [1] + \Delta [W] \), where [1] is the identity matrix and [W] is a matrix that you should construct explicitly and express in terms of the quantity \( \tan \theta = V/\Delta \). Show that these two matrices commute, and that any linear combination of the states |1\rangle and |2\rangle is an eigenstate of \( \epsilon_0 [1] \), with eigenvalue \( \epsilon_0 \). Thus, an eigenstate \( |\lambda\rangle = a|1\rangle + b|2\rangle \) of \( W \) with eigenvalue \( \lambda \) will be an eigenstate of \( H \) with eigenvalue \( \epsilon_0 + \lambda \Delta \).

(b) Solve the characteristic equation for \( [W] \) and show that its two eigenvalues can be written as \( \lambda_{\pm} = \pm \sec \theta \). Use this to find the energy eigenvalues \( \epsilon_{\pm} \) of \( H \). Express these eigenvalues in terms of \( \epsilon_0, \Delta, \) and \( V \). Plot the eigenvalues as a function of \( V/\Delta \) to demonstrate the phenomena of “level repulsion”.

(c) For \( \lambda_+ = \sec \theta \), solve the eigenvalue equation for \( W \) to obtain the corresponding eigenstate as a linear combination of the states |1\rangle and |2\rangle. Express your solution in terms of trigonometric functions of the half angle \( \theta/2 \).

(d) Do the same for \( \lambda_- = -\sec \theta \).