49. Let $H_0$ be a Hamiltonian with a degenerate spectrum of energies $\{\varepsilon_n^{(0)}\}$. Let the degeneracy of level $n$ be denoted by $N_n$, and let $S_n^{(0)}$ denote the $N_n$ dimensional eigenspace of $H_0$ associated with energy $\varepsilon_n^{(0)}$. A weak perturbation $H^{(1)}$ is applied. Briefly, but clearly describe the basic technique of degenerate perturbation theory, and describe what one obtains by applying that technique.

**Solution:** In degenerate perturbation theory, given an initial basis of states $\{\phi_n, \tau\} \mid \tau = 1, \ldots, N_n\}$, one constructs, in each $N_n$-dimensional eigenspace $S_n^{(0)}$ of $H_0$, the $N_n \times N_n$ matrix $[H_{n,n}^{(1)}]$ with elements

$$[H_{n,n}^{(1)}]_{\tau', \tau} = \langle \phi_n, \tau' | H^{(1)} | \phi_n, \tau \rangle$$

representing the perturbation within that eigenspace, and diagonalizes it (i.e., solves the eigenvalue problem for that matrix) to find (i) the first order energy shifts to energy level $\varepsilon_n^{(0)}$, which are identified with the eigenvalues of $[H_{n,n}^{(1)}]$, and (ii) a new basis $\{\chi_n, \tau\} \mid \tau = 1, \ldots, N_n\}$ of unperturbed eigenstates of $H^{(0)}$ that are not connected by the perturbing operator $H^{(1)}$, on which higher orders of perturbation can be computed without any diverging terms arising from vanishing energy denominators, since unperturbed states of the same energy are, in the new basis, not connected by the perturbation.
50. A two dimensional isotropic harmonic oscillator, expressed in appropriate dimensionless variables

\[ H^{(0)} = \sum_{i=x,y} \frac{\hbar \omega}{2} (\hat{q}_i^2 + \hat{p}_i^2) = \sum_{i=x,y} \hat{H}_i^{(0)} \]

is subject to a perturbation

\[ H^{(1)} = \lambda \hbar \omega \hat{q}_x \hat{q}_y. \]

(a) Find the unperturbed energy levels and the degeneracies of the unperturbed Hamiltonian, and express the eigenstates as appropriate direct product states formed from the 1d eigenstates of the implicitly defined operators \( H_x^{(0)} \) and \( H_y^{(0)} \).

The eigenstates of \( H^{(0)} \) will be of the form

\[ |n_x, n_y\rangle = |n_x\rangle^{(x)} |n_y\rangle^{(y)} \]

and will be eigenstates of \( H^{(0)} \) with energies

\[ \varepsilon^{(0)}_n = \left( n_x + \frac{1}{2} \right) \hbar \omega + \left( n_y + \frac{1}{2} \right) \hbar \omega = (n + 1) \hbar \omega \quad n = 0, 1, 2, \ldots \]

For a given level \( n \), we can have \( n_x = 0, 1, \ldots , n \) while \( n_y = n - n_x \). Thus the degeneracy of level \( n \) is

\[ N_n = n + 1. \]

(b) Find the first order energy level splitting, if any, of the lowest three (possibly degenerate) unperturbed energy levels of this system. Plot the corresponding energies as a function of the parameter \( \lambda \).

The ground state \( |0, 0\rangle \) is non-degenerate, and for this state,

\[ \Delta \varepsilon^{(1)}_0 = \lambda \hbar \omega \langle q_x | q_y \rangle = \lambda \hbar \omega \langle q_x \rangle^{(x)} \langle q_y \rangle^{(y)} = 0 \]

For the first excited state with \( n = 1 \), we have the two-fold degenerate states \(|0, 1\rangle \) and \(|1, 0\rangle \). Thus we have to diagonalize the sub-matrix \([H^{(1)}_{11}]\). Computing the matrix elements

\[ \langle m_x, m_y | H^{(1)} | n_x, n_y \rangle = \lambda \hbar \omega \langle m_x | q_x | n_x \rangle \langle m_y | q_y | n_y \rangle \]

\[ \frac{\lambda \hbar \omega}{2} [\sqrt{n_x + 1} \delta_{m_x, n_x + 1} + \sqrt{n_x} \delta_{m_x, n_x - 1}] [\sqrt{n_y + 1} \delta_{m_y, n_y + 1} + \sqrt{n_y} \delta_{m_y, n_y - 1}] \]

For the states of the \( n = 1 \) level this yields

\[ [H^{(1)}_{11}] = \begin{bmatrix} 0 & \frac{\lambda \hbar \omega}{2} \\ \frac{\lambda \hbar \omega}{2} & 0 \end{bmatrix} \]

Solving the characteristic equation \( \det[H^{(1)}_{11} - \varepsilon^{(1)}] = 0 \), we find first order energy shifts \( \varepsilon^{(1)}_1 = \pm \lambda \hbar \omega / 2 \), so these two degenerate states split into two nondegenerate states with energies

\[ \varepsilon_{1,+} = 2 \hbar \omega + \frac{\lambda \hbar \omega}{2} \]

\[ \varepsilon_{1,-} = 2 \hbar \omega - \frac{\lambda \hbar \omega}{2} \]

For the \( n = 2 \) level we have the three degenerate states \(|0, 2\rangle, |1, 1\rangle, \) and \(|2, 0\rangle \), which yields the \( 3 \times 3 \) matrix

\[ H^{(1)}_{22} = \begin{bmatrix} 0 & \frac{\lambda \hbar \omega}{2} & 0 \\ \frac{\lambda \hbar \omega}{2} & 0 & \frac{\lambda \hbar \omega}{2} \\ 0 & \frac{\lambda \hbar \omega}{2} & 0 \end{bmatrix} \]
Solving the characteristic equation we det\( [H^{(1)}_{22} - \varepsilon^{(1)}] = 0 \) we find first order energy shifts \( \varepsilon^{(1)} = 0 \), and \( \varepsilon^{(1)} = \pm \lambda \hbar \omega \), so these three states split into three nondegenerate states with energies

\[
\begin{align*}
\varepsilon^{1}_{2,+} &= 3\hbar \omega + \lambda \hbar \omega \\
\varepsilon^{1}_{2,0} &= 3\hbar \omega \\
\varepsilon^{1}_{2,-} &= 3\hbar \omega - \lambda \hbar \omega 
\end{align*}
\]

(c) For the first excited state, find a new basis of eigenstates for that unperturbed level that are not connected by the perturbation, and show that at least some of them are not direct product states. Substituting back into the eigenvalue equation for \( [H^{(1)}_{11}] \) for eigenvalue \( \pm \lambda \hbar \omega / 2 \) we obtain

\[
[H^{(1)}_{11}] [\chi] = \begin{bmatrix}
\mp \lambda \hbar \omega / 2 & \lambda \hbar \omega / 2 \\
\pm \lambda \hbar \omega / 2 & \mp \lambda \hbar \omega / 2
\end{bmatrix} \begin{bmatrix}
\chi^+_1 \\
\chi^+_2
\end{bmatrix} = 0
\]

Dividing this by \( \hbar \omega \) gives the equations

\[
\begin{align*}
-\chi^+_1 + \chi^+_2 &= 0 \\
\chi^-_1 + \chi^-_2 &= 0
\end{align*}
\]

which after normalization gives the states

\[
\begin{align*}
|\varepsilon_{1,+} \rangle &= \frac{1}{\sqrt{2}} \left( |0, 1 \rangle + |1, 0 \rangle \right) \\
|\varepsilon_{1,-} \rangle &= \frac{1}{\sqrt{2}} \left( |0, 1 \rangle - |1, 0 \rangle \right)
\end{align*}
\]

which cannot be factored into direct product states. Thus motion along the \( x \) and \( y \) directions has become entangled.
51. A system with Hamiltonian $H^{(0)}$ is subject to a perturbation $H^{(1)}$, which in a certain ONB can be represented by the following matrices

$$
[H^{(0)}] = \begin{pmatrix}
\varepsilon_0 & 0 & 0 & 0 & 0 \\
0 & \varepsilon_0 & 0 & 0 & 0 \\
0 & 0 & \varepsilon_0 & 0 & 0 \\
0 & 0 & 2\varepsilon_0 & 0 & 0 \\
0 & 0 & 0 & 2\varepsilon_0 & 0
\end{pmatrix}
$$

$$
[H^{(1)}] = \begin{pmatrix}
0 & \Delta & 0 & 3\Delta & 0 \\
\Delta & 0 & 0 & 0 & 0 \\
0 & 0 & 4\Delta & 0 & 0 \\
3\Delta & 0 & 0 & 0 & 2i\Delta \\
0 & 0 & 0 & -2i\Delta & 0
\end{pmatrix}
$$

(a) Find the new energies of this system, correct to first order in the perturbation.

(b) Find a new basis of eigenstates of $H^{(0)}$ such that states of the same unperturbed energy are not connected by $H^{(1)}$.

**Solution:** We will do both parts a) and b) simultaneously.

According to the basic principle of degenerate perturbation theory, we must diagonalize the perturbation within each degenerate subspace. In the 3-fold degenerate subspace associated with energy $\varepsilon_0$ we have the matrix

$$
[H^{(1)}]^{\varepsilon_0} = \begin{pmatrix}
0 & \Delta & 0 \\
\Delta & 0 & 0 \\
0 & 0 & 4\Delta
\end{pmatrix}.
$$

Solving the characteristic equation $\det \left( (H^{(1)})^{\varepsilon_0} - \varepsilon \right) = 0 = (\varepsilon^2 - \Delta^2) \left( \varepsilon - 4\Delta \right)$, gives

$$
\varepsilon_0^{(1)} = 4\Delta \\
\varepsilon_+^{(1)} = +\Delta \\
\varepsilon_-^{(1)} = -\Delta
$$

which completely lifts the degeneracy associated with the $\varepsilon_0$ level, i.e.,

$$
\varepsilon_0 \rightarrow \begin{array}{c}
\varepsilon_0 + 4\Delta \\
\varepsilon_0 + \Delta \\
\varepsilon_0 - \Delta
\end{array}
$$

For the eigenstate of $[H^{(1)}]$ with eigenvalue $\varepsilon_0^{(1)} = 4\Delta$ one has

$$
[H^{(1)} - \varepsilon_0^{(1)}][\chi] = \begin{pmatrix}
-4\Delta & \Delta & 0 \\
\Delta & -4\Delta & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\chi_1 \\
\chi_2 \\
\chi_3
\end{pmatrix} = 0
$$

from which we obtain the equations

$$
-4\Delta \chi_1 + \Delta \chi_2 = 0 \quad \text{so} \quad \chi_2 = 4\chi_1 \\
\Delta \chi_1 - 4\Delta \chi_2 = 0 \quad \text{so} \quad \chi_1 = 4\chi_2
$$

The last equation gives $0 + 0 + 0 = 0$ and gives us no information about the coefficients. Substituting the result of the first equation into the second one finds that $\chi_1 = 16\chi_1$, so that $15\chi_1 = 0$, which implies that $\chi_1 = 0$, and so $\chi_2 = 4\chi_1 = 0$. For this state not to be the null vector (which is never an eigenstate even though it always satisfies the eigenvalue equation) we must have $\chi_3$ not vanish. Normalization then requires that $\chi_3 = 1$, so that $|\varepsilon_0 + 4\Delta\rangle = 0|\!\!\!1\rangle + 0|\!\!\!2\rangle + 1|\!\!\!3\rangle = |\!\!\!3\rangle$.

$$
|\varepsilon_0 + 4\Delta\rangle = |\!\!\!3\rangle \\
|\varepsilon_0 + \Delta\rangle = \frac{1}{\sqrt{2}} [|\!\!\!1\rangle + |\!\!\!2\rangle] \\
|\varepsilon_0 - \Delta\rangle = \frac{1}{\sqrt{2}} [|\!\!\!1\rangle - |\!\!\!2\rangle]
$$
which diagonalize the submatrix for $H^{(1)}$ in this subspace.

In the 2-fold degenerate subspace associated with energy $2\varepsilon_0$ we have the matrix

$$
\begin{pmatrix}
0 & 2i\Delta \\
-2i\Delta & 0
\end{pmatrix}.
$$

Setting $\det \left( [H^{(1)}]^{2\varepsilon_0} - \varepsilon \right) = 0 = (\varepsilon^2 - 4\Delta^2)$, gives

$$
\varepsilon_4^{(1)} = 2\Delta,
$$

$$
\varepsilon_5^{(1)} = -2\Delta.
$$

which lifts the degeneracy associated with this level. The final eigenstates and energy eigenvalues in this subspace of $H_0$ to this order are:

$$
\varepsilon = 2(\varepsilon_0 + \Delta), \quad |2(\varepsilon_0 + \Delta)\rangle = \frac{1}{\sqrt{2}} \left[ |4\rangle - i|5\rangle \right]
$$

$$
\varepsilon = 2(\varepsilon_0 - \Delta), \quad |2(\varepsilon_0 - \Delta)\rangle = \frac{1}{\sqrt{2}} \left[ |4\rangle + i|5\rangle \right]
$$

(c) Find the energy of the new ground state to 2nd order in the perturbation. The new ground state is

$$
|\varepsilon_0 - \Delta\rangle = \frac{1}{\sqrt{2}} \left[ |1\rangle - |2\rangle \right]
$$

The perturbing operator has non-zero matrix elements connecting this state to other members of the new basis set:

$$
\langle \varepsilon_0 - \Delta | H^{(1)} | 2(\varepsilon_0 + \Delta) \rangle = \frac{1}{\sqrt{2}} \left[ \langle 1 | - \langle 2 | \right] H^{(1)} \frac{1}{\sqrt{2}} \left[ |4\rangle - i|5\rangle \right]
$$

$$
= \frac{1}{2} \langle 1 | H^{(1)} | 4 \rangle = \frac{3}{2} \Delta
$$

$$
\langle \varepsilon_0 - \Delta | H^{(1)} | 2(\varepsilon_0 - \Delta) \rangle = \frac{1}{\sqrt{2}} \left[ \langle 1 | - \langle 2 | \right] H^{(1)} \frac{1}{\sqrt{2}} \left[ |4\rangle + i|5\rangle \right]
$$

$$
= \frac{1}{2} \langle 1 | H^{(1)} | 4 \rangle = \frac{3}{2} \Delta
$$

Using our formula for non-degenerate perturbation theory we find

$$
\Delta \varepsilon_0^{(2)} = -\left[ \frac{9}{4} \Delta^2 + \frac{9}{4} \Delta^2 \right] = -\frac{9}{2} \Delta^2 \varepsilon_0,
$$

thus, correct to 2nd order we find the ground state energy

$$
E_0 = \varepsilon_0 \left( 1 - \frac{\Delta}{\varepsilon_0} - \frac{9}{2} \frac{\Delta^2}{\varepsilon_0^2} \right).
$$
52. Briefly but clearly state the four postulates (feel free to look these up) of the general formulation of quantum mechanics as they apply to arbitrary quantum mechanical systems (e.g., to the universe itself) and explain the differences (where they arise) between those postulates and the postulates of Schrödinger’s wave mechanics for a single particle.

Does the universe have a wave function, a state vector, both, neither? Explain. (no wrong answer here, just interested in what you think.)