4.1 Fibonacci

Tuesday, September 12, 2023 11:39 AM

Famos sequence of numbes · Filio - naccio

$0, 1, 1, 2, 3, 5, 8, 13, \ldots$

formula

- $F(n) = F(n-1) + F(n-2)$ ← recurrance.
- $F(0) = 0$
- $F(1) = 1$

} initial conditions.

```
FUNCTION Fibo(n)
  IF n ≤ 1 RETURN n
  ELSE
     RETURN Fibo(n-1) + Fibo(n-2)
```

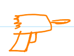
basic operation +

count number of basic operations

- $A(n) = A(n-1) + 1 + A(n-2)$ ← recurrance
- $A(0) = 0$
- $A(1) = 0$

} initial conditions

NOTE: instead of Backwards substitution:
I will introduce power tools.



• TOOLS: Results from mathematics

• DEF: Linear Second-Order recurrances with Constant coefficients.
recurrances of the form

$$a \cdot f(n) + b \cdot f(n-1) + c \cdot f(n-2) = g(n)$$

where a, b, c are real numbers, $a \neq 0$

second order: $f(n-2)$ is two steps removed from $f(n)$

two types:
- homogeneous: $g(n) = 0$ for all "n"
- inhomogeneous

• DEF: the characteristic equation of an homogeneous Linear second-order recurrance with constant coefficients is:
the cuadratic:

$$a \cdot r^2 + b \cdot r + c = 0$$

E.G.

$$f(n) - 6 \cdot f(n-1) + 9 \cdot f(n-2) = 0$$

$$\Downarrow$$

$$r^2 - 6 \cdot r + 9 = 0$$

$a = 1$
$b = -6$
$c = 9$

• THEOREM: Let r_1, r_2 be the roots of a characteristic equation:

Case 1: r_1 r_2 are real and distinct,
then the solution to the recurrance is

$$f(n) = \alpha \cdot r_1^n + \beta \cdot r_2^n$$

where α, β are real constants.

Case 2: r_1 r_2 are equal
then the solution to the reccurance is

$$f(n) = \alpha \cdot r^n + \beta \cdot n \cdot r^n$$

where α β are real constants.

Case 3: r_1 r_2 are distinct complex numbers.
$r_{1,2} = u \pm i \cdot v$
then the solution to the reccurance is

$$f(n) = \gamma^n \cdot [\alpha \cdot \cos(n \cdot \theta) + \beta \cdot \sin(n \cdot \theta)]$$

where $\gamma = \sqrt{u^2 + v^2}$, $\theta = \arctan v/u$, α, β are treal constants.

End tools:

Back to Fibonacci:

- $F(n) = F(n-1) + F(n-2)$
- $F(0) = 0$
- $F(1) = 1$

$$F(n) - F(n-1) - F(n-2) = 0$$

← second order reccurance w/ constant coefficients.

$a = 1$
$b = -1$
$c = -1$

So: the characteristic equation is:

$$r^2 - r - 1 = 0$$

with roots:

$$r_{1,2} = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$F(n) = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

← from case 1 of ovobe theorem

$$F(0) = 0 = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^0 + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha + \beta = 0$$

- $\alpha = -\beta$
- $\beta = -\alpha$

- $F(1) = 1 = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$

We can now solve for α, β

$$\alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right) - \alpha\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\alpha \cdot \left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = 1$$

$$\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5} - 1 + \sqrt{5}}{2} = \frac{\cancel{2} \cdot \sqrt{5}}{\cancel{2}}$$

$$\alpha \cdot [\sqrt{5}] = 1$$

$$\alpha = \frac{1}{\sqrt{5}}$$

$$\alpha = \frac{1}{\sqrt{5}} \qquad \beta = -\frac{1}{\sqrt{5}}$$

$\alpha = \frac{1}{\sqrt{5}} \qquad \beta = -\frac{1}{\sqrt{5}}$

$$F(n) = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F(n) = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) \quad \text{where} \quad \phi = \left(\frac{1+\sqrt{5}}{2}\right) \approx 1.61803\ldots$$

the golden ratio.

$$\hat{\phi} = -1/\phi \approx 0.61803\ldots$$

Note:
this closed formula tells that Fibonacci grows exponentialy

$$F(n) \in \Theta(\phi^n)$$

Note:

$F(n) = \frac{1}{\sqrt{5}}\phi^n$ rounded to the nearest integer.

- BACK TO THE RECURSIVE ALGORITHM FOR FIBONACCI

count number of basic operations

- $A(n) = A(n-1) + 1 + A(n-2)$ ← recurrance
- $A(0) = 0$
- $A(1) = 0$

} initial conditions

$$A(n) - A(n-1) - A(n-2) = 1$$

Let $B(n) = A(n) + 1$

- $[A(n)+1] - [A(n-1)+1] - [A(n-2)+1] = 0$
- $B(n) - B(n-1) - B(n-2) = 0.$

the characteristic equation:

$r^2 - r - 1 = 0$ Same as fibonacci

Note: $B(n) = F(n+1)$

So.

$$B(n) = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right)$$

$$A(n) = B(n) - 1$$

$$A(n) = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right) - 1$$

Then

$$A(n) \in \Theta(\phi^n)$$

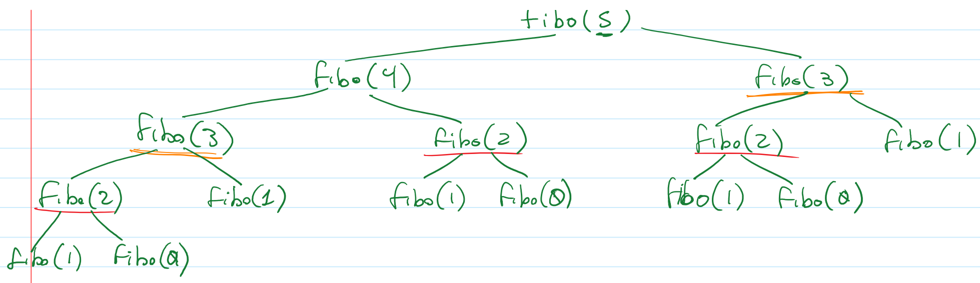
- ANOTHER VIEW:

```
FUNCTION Fibo(n)
    IF n ≤ 1 RETURN n
    ELSE
        RETURN Fibo(n-1) + Fibo(n-2)
```

Call Tree:

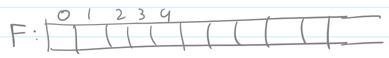
fibo(5)

fibo(4) fibo(3)



[recursion is powerfull, but it's succinctess may hide inefficiencies.]

```
FUNCTION  FibArray ( F )
    F[0] ← 0
    F[1] ← 1
    FOR i←2 to n DO
        F[i] ← F[i-1] + F[i-2]
    RETURN F[n]
```



$$A(n) = \sum_{i=2}^{n} 1 = n-2 \in \Theta(n)$$

EXTRA:
There is a $\Theta(\log n)$ way to compute fibonacci based on:

$$\begin{bmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$$

and an efficient way to compute matrix powers.

— EOF —