4.1 Fibonacci

Thursday, September 12, 2024 11:32 AM

Famous sequence of numbers

0 1 1 2 3 5 8 13 21 34 .....

Formula

$F(n) = F(n-1) + F(n-2)$ ← recurrance

$F(0) = 0$

$F(1) = 1$

} initial conditions.

Pseudocode

```
FUNCTION fibo(n)
  if n ≤ 1 return n
  else
    return fibo(n-1) + fibo(n-2)
```

Analysis.

basic operation

count number of additions

- $A(n) = A(n-1) + 1 + A(n-2)$ ← recurrance
- $A(0) = 0$
- $A(1) = 0$

} initial conditions.

Note: very difficult to solve using backwards substitution

need to introduce MORE tools:

- TOOLS: Linear Second-Order Recurrences

DEF: linear Second-Order Recurrences are recurrences of the form:

$$a \cdot f(n) + b \cdot f(n-1) + c \cdot f(n-2) = g(n)$$

where a, b c are real numbers. $a \neq 0$

"Second-order": $f(n-2)$ is two steps behind $f(n)$

two types of linear second order recurrances

→ homogeneous if $g(n) = 0$ for all n

→ inhomogeneous.

DEF: the characteristic equation of an homogeneous linear second-order recurrance is.

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$$a \cdot r^2 + b \cdot r + c = 0$$

E.g.

$$f(n) - 6f(n-1) + 9f(n-2) = 0$$

$a = 1$
$b = -6$
$c = 9$

the characteristic equation is:

$$1 \cdot r^2 - 6r + 9 = 0$$

THEOREM: from the roots of the char. eq. we can solve the recurrance.

Let r_1 r_2 be the roots of a characteristic equation:

Case 1: r_1 r_2 are real and distinct:
then the solution for the recurrance is:

$$f(n) = \alpha \cdot r_1^n + \beta \cdot r_2^n$$

where α β are some real numbers.

Case 2: r_1 r_2 are equal
then the solution for the reccurance is

$$f(n) = \alpha \cdot r^n + \beta \cdot n \cdot r^n$$

where α β are some rea numbers.

Case 3: r_1 r_2 are distinct complex numbers
then the solution for the reccurance is

$$r_{1,2} = u \pm i \cdot v$$

$$f(n) = \gamma^n \cdot [\alpha \cdot \cos(n \cdot \theta) + \beta \cdot \sin(n \cdot \theta)]$$

where

$\gamma = \sqrt{u^2 + v^2}$ $\quad \theta = \arctan\left(\frac{v}{u}\right)$ $\quad \alpha, \beta$ some real numbers

QUADRATIC: REVIEW:

- $a \cdot r^2 + b \cdot r + c$
- $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Back to Fibonacci:

Formula

$F(n) = F(n-1) + F(n-2)$ ← recurrance

Back to Fibonacci:

Formula

$$F(n) = F(n-1) + F(n-2) \leftarrow \text{recurrance}$$

$$\left.\begin{array}{l} F(0) = 0 \\ F(1) = 1 \end{array}\right\} \text{initial conditions.}$$

$$F(n) - F(n-1) - F(n-2) = 0$$

$a = 1$
$b = -1$
$c = -1$

Linear
second order recurrance.
w/ constant coefficients
Homogeneous.

Characteristic equation

$$r^2 - r - 1 = 0$$

with roots:

$$r_{1,2} = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

then:

$$F(n) = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

from case 1

Now lets find α β

- $F(0) = 0 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^0 + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha + \beta = 0$
- $\alpha = -\beta$ • $\beta = -\alpha$

$$F(1) = 1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^1$$

substitute β

$$1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^1 - \alpha\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$1 = \alpha\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right]$$

$$1 = \alpha\left[\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right]$$

$$1 = \alpha\left[\frac{\cancel{2}\cdot\sqrt{5}}{\cancel{2}}\right]$$

$$1 = \alpha\left[\sqrt{5}\right]$$

$$\alpha = \frac{1}{\sqrt{5}} \qquad \beta = -\frac{1}{\sqrt{5}}$$

So: closed Formula:

$\alpha = \frac{1}{\sqrt{5}} \qquad \beta = -\frac{1}{\sqrt{5}}$

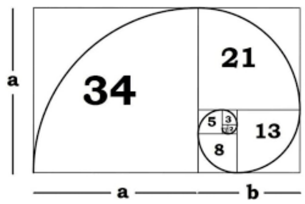
So: closed Formula:

$$F(n) = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

note

$$\left(\frac{1+\sqrt{5}}{2}\right) = 1.61803.... = \phi$$ "the golden ratio"

Let $\hat{\phi} = \frac{-1}{\phi}$



then:

$$F(n) = \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n\right)$$

$F(n) \in O(\phi^n)$ exponential.

Side Note.

you can compute $F(n) \cong \frac{1}{\sqrt{5}} \cdot \phi^n$ rounded to nearest integer.

Back to the recursive algorithm Fibo():

- $A(n) = A(n-1) + 1 + A(n-2)$ ← recurrance
- $A(0) = 0$
- $A(1) = 0$

} initial conditions.

- $A(n) - A(n-1) - A(n-2) = 1$

- Linear, second order recurrance
- constant coefficients,
- inhomogeneous.

Trick: rewrite formula:

- $[A(n)+1] - [A(n-1)+1] - [A(n-2)+1] = 0$

Let $B(n) = A(n) + 1$

- $B(n) - B(n-1) - B(n-2) = 0$ - homogeneous!!

$a = 1$
$b = -1$
$c = -2$

$$r^2 - r - 1 = 0$$ ← same as fibonacci.

- $B(0) = 1$
- $B(1) = 1$
- $B(2) = 2$
- $B(3) = 3$
- $B(4) = 5$
- $B(5) = 8$

$$B(n) = F(n+1)$$

$$B(n) = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right)$$

$$A(n) - 1 = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right)$$

$$A(n) = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right) - 1$$

Then

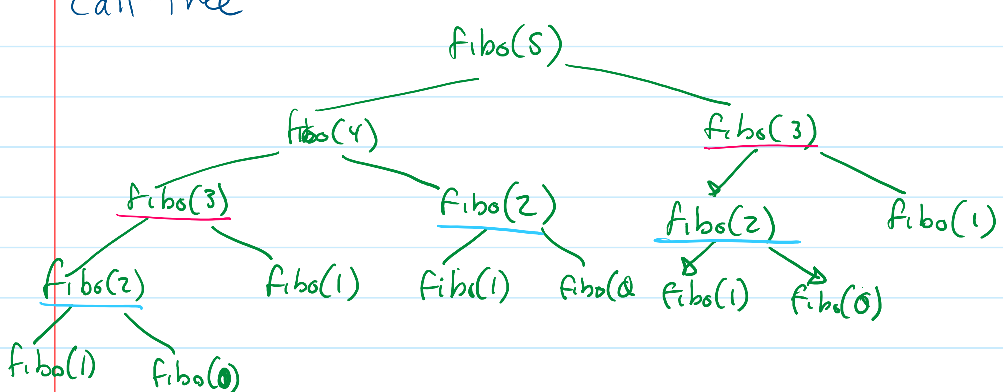
$$A(n) \in \Theta(\phi^n)$$

Another View: The call tree of fibo() :

Pseudocode

```
FUNCTION fibo(n)
   If n ≤ 1 return n
   else
      return fibo(n-1) + fibo(n-2)
```

Call-Tree



recursion is powerfull, but it's succinctess may hide inefficiencies recursion is powerfull, but it's succinctess may hide inefficiencies

Let's rewrite Fibo()

FUNCTION FibArray (F[0...n])

  F[0] ← 0

  F[1] ← 1

  for i ← 2 to n do

    F[i] ← F[i-1] + F[i-2]

  return F

$$F \; \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \multicolumn{9}{c}{} & \multicolumn{1}{c}{n} \\ \hline 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \cdots & \\ \hline \end{array}$$

Analysis:

basic-operation

$$A(n) = \sum_{i=2}^{n} 1 = n-2 \in \Theta(n) \quad \text{linear.}$$

EXTRA:

There is an $\Theta(\log n)$ way to compute fibonacci based on:

$$\begin{bmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$$

and an efficient way to compute powers of matrices.

— EOF —