4.1 Fibonacci

Thursday, February 20, 2025 12:34 PM

Famous sequence of numbers

0 1 1 2 3 5 8 13 21 34 55 .....

Formula

$$F(n) = F(n-1) + F(n-2)$$ – recurrance.

$$F(0) = 0$$
$$F(1) = 1$$

} initial conditions

Pseudocode

```
FUNCTION fibo(n)
  IF n ≤ 1 THEN
     RETURN n
  ELSE
     RETURN fibo(n-1) + fibo(n-2)
```

Analyse:

basic_operation. +

Count the number of additions.

$$A(n) = A(n-1) + 1 + A(n-2)$$ – recurrence

$$A(0) = 0$$
$$A(1) = 0$$

} initial conditions

Note: not easy to solve using backward substitution

So we need to introduce more tools.

- TOOLS : Linear Second-Order Recurrences

- DEF: linear Second-order Recurrences are Recurrences of the form:

$$a \cdot f(n) + b \cdot f(n-1) + c \cdot f(n-2) = g(n)$$

where: a b c are real numbers

$a \neq 0$

"second order" $f(n-2)$ is "two steps" behind $f(n)$

two types → homogeneous if $g(n) = 0$ for all n
→ inhomogeneous

- DEF: the characteristic equation of an homogeneous linear second-order recurrence is:

$$a \cdot r^2 + b \cdot r + c = 0 \quad (r \text{ is the variable})$$

E.g. $a = 13$, $b = 1$, $c = 3$. $13 \cdot f(n) + 1 \cdot f(n-1) + 3 \cdot f(n-2) = 0$

the characteristic equation is: $13 \cdot r^2 + 1 \cdot r + 3 = 0$

- THEOREM: from the roots of a characteristic equation we can solve a homogeneous linear second-order recurrence.

Let r_1 r_2 be the roots of a characteristic equation:

Case 1: r_1 r_2 are real and distinct.
then the solution for the recurrence is

$$f(n) = \alpha \cdot r_1^n + \beta \cdot r_2^n$$

where α β are some real numbers.

Case 2: r_1 r_2 are equal
then the solution for the recurrence is

$$f(n) = \alpha \cdot r^n + \beta \cdot n \cdot r^n$$

where α β are some real numbers.

Case 3: r_1 r_2 are distinct complex numbers
then the solution for the recurrance is

$$r_{1,2} = u \pm i \cdot v$$

$$f(n) = \gamma^n \cdot [\alpha \cdot \cos(n \cdot \theta) + \beta \cdot \sin(n \cdot \theta)]$$

where:
$\gamma = \sqrt{u^2 + v^2}$ $\quad \theta = \arctan\left(\frac{v}{u}\right)$ $\quad \alpha$ β some real numbers

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- REVIEW: Quadratic Formula
  - $a \cdot r^2 + b \cdot r + c$
  - $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

- Back to Fibonacci:

Formula

$F(n) = F(n-1) + F(n-2)$ — recurrance.

$F(0) = 0$
$F(1) = 1$
} initial conditions

- $F(n) - F(n-1) - F(n-2) = 0$ homogeneous linear second-order recurrence.

$a = 1$
$b = -1$
$c = -1$

characteristic equation:

$$1 \cdot r^2 - 1 \cdot r - 1 = 0$$

with roots:

$$r_{1,2} = \frac{1 \pm \sqrt{1-4(1 \cdot -1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

then: from Case 1:

$$F(n) = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Now: Lets find α β:

- $F(0) = 0 = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^0 + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^0$

$$0 = \alpha + \beta$$

- $\alpha = -\beta$ $\quad$ • $\beta = -\alpha$

- $F(1) = 1 = \alpha \cdot \left(1+\sqrt{5}\right)^1 + \beta \cdot \left(1-\sqrt{5}\right)^1$

$$\bullet\ F(1) = 1 = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta \cdot \left(\frac{1-\sqrt{5}}{2}\right)^1$$

Substitute β

$$F(1) = 1 = \alpha \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 - \alpha\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$1 = \alpha\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right]$$

$$= \alpha\left[\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right]$$

$$= \alpha\left[\frac{\cancel{2}\cdot\sqrt{5}}{\cancel{2}}\right]$$

$$1 = \alpha\left[\sqrt{5}\right]$$

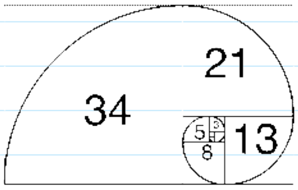
$$\alpha = \frac{1}{\sqrt{5}} \qquad \beta = -\frac{1}{\sqrt{5}}$$

So: closed formula

$$F(n) = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

note:

$$\frac{1+\sqrt{5}}{2} = 1.61803\ldots\ldots = \phi$$ "the golden ratio"



Let $\hat{\phi} = \dfrac{-1}{\phi}$

Then:

$$F(n) = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) \in O(\phi^n)$$

Side Note:

You can compute $F(n) \simeq \frac{1}{\sqrt{5}} \cdot \phi^n$ rounded to

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You can compute $F(n) \cong \frac{1}{\sqrt{5}} \cdot \phi^n$ rounded to nearest integer.

- Back to the recursive algorithm fibo()

$\cdot\ A(n) = A(n-1) + 1 + A(n-2)$ — recurrence

$\cdot\ A(0) = 0$
$\cdot\ A(1) = 0$
} initial conditions

$\cdot\ A(n) - A(n-1) - A(n-2) = \underline{1}$ • it's inhomogeneous

Trick:

Rewrite formula to make it homogeneous.

$\cdot\ [A(n)+1] - [A(n-1)+1] - [A(n-2)+1] = 0$

$A(n) \cancel{+1} - A(n-1) \cancel{-1} - A(n-2) - 1 = 0$

$A(n) - A(n-1) - A(n-2) = 1$

Let $B(n) = A(n) + 1$

$\cdot\ B(n) - B(n-1) - B(n-2) = 0$

coefficients
$a = 1$
$b = -1$
$c = -1$

characteristic equation: $ar^2 + br + c$

$r^2 - r - c = 0$

But, Let's take a shortcut

$B(0) = A(0) + 1 = 1$
$B(1) = A(1) + 1 = 1$
$B(2) = 2$
$B(3) = 3$
$B(4) = 5$
$B(5) = 8$
$B(6) = 13$
$B(7) = 21$

$B(n) = B(n-1) + B(n-2)$

$B(n) = F(n+1)$

$B(6) = 13$
$B(7) = 21$ $\qquad B(n) = F(n+1)$
$\vdots$

So:

$$B(n) = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right)$$

$$A(n) - 1 = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right)$$

$$A(n) = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right) - 1$$

Then

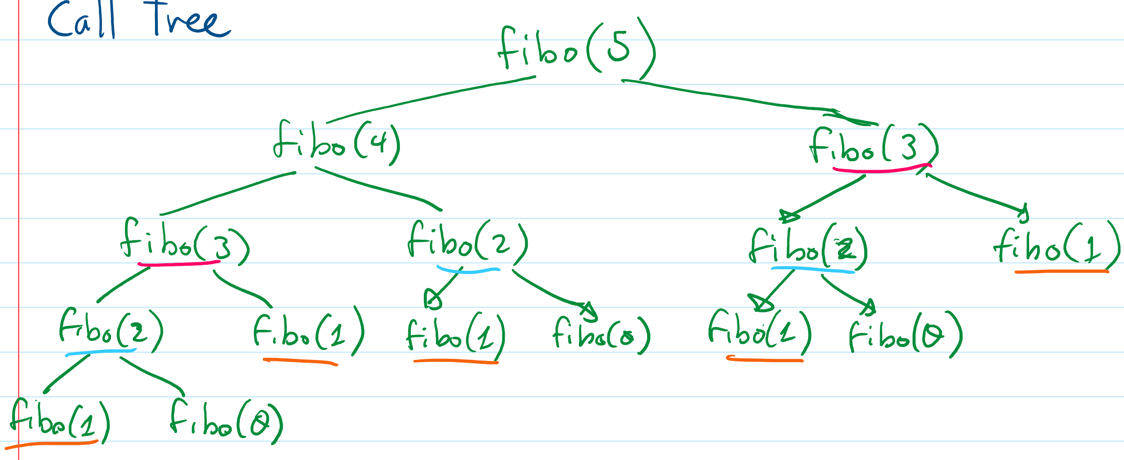
$$A(n) \in \Theta(\phi^n)$$

- ANOTHER VIEW : The call tree of fibo()

Pseudocode

```
FUNCTION fibo(n)
  IF n ≤ 1 THEN
     RETURN n
  ELSE RETURN fibo(n-1) + fibo(n-2)
```

Call Tree



recursion <u>is</u> powerfull, but it's succinctness may hide inefficiencies.

Let's rewrite fibo()

```
FUNCTION FiboArray ( F[0...n] )
    F[0] ← 0
    F[1] ← 1
    FOR i ← 2 TO n DO
        F[i] ← F[i-1] + F[i-2]
    RETURN F
```

0	1	2	3	4	5	6	7	8	9	
0	1	1	2	3	5	8	13	21	34	...

Analysis

basic operation $+$

$$A(n) = \sum_{i=2}^{n} 1 = n-2 \in \Theta(n)$$ linear.

EXTRA:

There is a $\Theta(\log n)$ way to compute fibonacci, based on:

$$\begin{bmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n$$

and an efficient way to compute powers of matrices

—o— EOF.