

PROJECTIVE FRAÏSSÉ LIMITS OF TREES

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ABSTRACT. We continue study of projective Fraïssé limit developed by Irvin, Panagiotopoulos and Solecki. We modify the ideas of monotone, confluent, retraction and light mappings from continuum theory as well as several properties of continua so as to apply to topological graphs. As the topological realizations of the Fraïssé limits we obtain either some known continua, for example the dendrite D_3 or the Cantor fan, or quite new, interesting ones for which we do not yet have topological characterizations.

1. INTRODUCTION AND DEFINITIONS

In [7], T. Irvin and S. Solecki introduced the idea of a projective Fraïssé limit as a dualization of the injective Fraïssé limit from model theory. In that paper they construct the pseudo arc as the topological realization of a projective Fraïssé limit of a certain class of finite graphs and epimorphisms between members of the class. Subsequently, D. Bartošová and A. Kwiatkowska, [2], Kubiś and A. Kwiatkowska, [8] and A. Panagiotopoulos and S. Solecki, [11], extended these ideas to study, respectively, the Lelek fan, the Lelek fan and the Poulsen simplex, and the Menger curve as Fraïssé limits.

In this article we continue this study by considering families of finite trees and epimorphisms (mappings) between them that satisfy certain properties. We modify the ideas of monotone, confluent, retraction and light mappings from continuum theory as well as several properties of continua so as to apply to topological graphs. Tools are developed for studying projective Fraïssé limits and their topological realizations. As the topological realizations of the Fraïssé limits we obtain several known continua, for example the dendrite D_3 or the Cantor fan. However, in the cases where epimorphisms are restricted to being confluent or confluent and end-preserving we have continua that have many interesting properties and which may be different from any known continua.

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A *graph* is an ordered pair $A = (V(A), E(A))$, where $E(A) \subseteq V(A)^2$ is a reflexive and symmetric relation on $V(A)$. The elements of $V(A)$ are called *vertices* of graph A and elements of $E(A)$ are called *edges*.

A *topological graph* K is a graph $(V(K), E(K))$, whose domain $V(K)$ is a 0-dimensional, compact, second-countable (thus metric) space and $E(K)$ is a closed, reflexive and symmetric subset of $V(K)^2$. A topological graph is an example of topological \mathcal{L} structure. For a general definition of a topological \mathcal{L} structure see [7].

Given two graphs A and B a function $f: V(A) \rightarrow V(B)$ is a homomorphism if it maps edges to edges, i.e. $\langle a, b \rangle \in E(A)$ implies $\langle f(a), f(b) \rangle \in E(B)$. If we consider rooted trees, then we have an additional relation, namely an order \leq on vertices, and homomorphisms have to preserve the order. Details are in Section 5. A homomorphism f is an epimorphism if it is moreover surjective on both vertices and edges. An isomorphism is an injective epimorphism.

Definition 1.1. A finite graph T is a *tree* if for every two distinct vertices $a, b \in T$ there is a unique finite sequence $v_0 = a, v_1, \dots, v_n = b$ of vertices in T such that for every $i \in \{0, 1, \dots, n-1\}$ we have $\langle v_i, v_{i+1} \rangle \in E(T)$ and $v_i \neq v_{i+1}$. Let n be a natural number, a vertex $p \in T$ has order n ($\text{ord}(p) = n$) if there are n non-degenerate edges in T that contain p . If $\text{ord}(p) = 1$ then p is an end vertex, $\text{ord}(p) = 2$ then p is an ordinary vertex and if $\text{ord}(p) \geq 3$ then p a ramification vertex.

The theory of projective Fraïssé limits were developed in [7] and further refined in [11]. We literally recall their definitions here.

Definition 1.2. Let \mathcal{F} be a class of finite graphs with a fixed family of morphisms among the structures in \mathcal{F} . We assume that each morphism is an epimorphism with respect to \mathcal{F} . We say that \mathcal{F} is a projective Fraïssé class if

- (1) \mathcal{F} is countable up to isomorphism, that is, any sub-collection of pairwise non-isomorphic structures of \mathcal{F} is countable;
- (2) morphisms are closed under composition and each identity map is a morphism;
- (3) for $B, C \in \mathcal{F}$ there exist $D \in \mathcal{F}$ and morphisms $f: D \rightarrow B$ and $g: D \rightarrow C$; and
- (4) for every two morphisms $f: B \rightarrow A$ and $g: C \rightarrow A$, there exist morphisms $f_0: D \rightarrow B$ and $g_0: D \rightarrow C$ such that $f \circ f_0 = g \circ g_0$, i.e. the diagram (D1) commutes.

(D1)

$$\begin{array}{ccc}
 & B & \\
 f \swarrow & & \dashrightarrow f_0 \\
 A & & D \\
 g \swarrow & & \dashrightarrow g_0 \\
 & C &
 \end{array}$$

We will refer to the last property above as the projective amalgamation property.

The class \mathcal{F} of finite graphs and epimorphisms is enlarged to a class \mathcal{F}^ω which includes all topological graphs obtained as inverse limits of graphs in \mathcal{F} with bonding maps from the family of epimorphisms. If $G = \varprojlim \{G_n, \alpha_n\} \in \mathcal{F}^\omega$ and $a = (a_n)$ and $b = (b_n)$ are elements of G then $\langle a, b \rangle$ is an edge in G if and only if for each n , $\langle a_n, b_n \rangle$ is an edge in G_n . An epimorphism h between a topological graph $G = \varprojlim \{G_n, \alpha_n\}$ in \mathcal{F}^ω and a finite graph $A \in \mathcal{F}$ is in the class \mathcal{F}^ω if and only if there is an m and an epimorphism $h' : G_m \rightarrow A$, $h' \in \mathcal{F}$, such that $h = h' \circ \alpha_m^\infty$ where α_m^∞ is the canonical projection from the inverse limit space onto the m th factor space. Finally, if K and L are inverse limit spaces in \mathcal{F}^ω an epimorphism $h : L \rightarrow K$ is in the family \mathcal{F}^ω if and only if for any finite graph $A \in \mathcal{F}$ and any epimorphism $g : K \rightarrow A$ in \mathcal{F}^ω , $g \circ h \in \mathcal{F}^\omega$.

In the proof of [7, Theorem 2.4] an inverse sequence satisfying certain properties, see definition below, was introduced and used to show the existence and uniqueness of the projective Fraïssé limit.

Definition 1.3. Given a projective family \mathcal{F} an inverse sequence $\{F_n, \alpha_n\}$ where $F_n \in \mathcal{F}$ and $\alpha_n : F_{n+1} \rightarrow F_n$ are epimorphisms in \mathcal{F} is said to be a *fundamental sequence* for \mathcal{F} if the following two conditions hold.

- (1) For any $G \in \mathcal{F}$ there is an n and an epimorphism from F_n onto G ;
- (2) For any n , any pair $G, H \in \mathcal{F}$, and any epimorphisms $g : H \rightarrow G$ and $f : F_n \rightarrow G$ there exists $m > n$ and an epimorphism $h : F_m \rightarrow H$ such that $g \circ h = f \circ \alpha_n^m$ i.e the diagram (D2) commutes.

(D2)

$$\begin{array}{ccc}
 F_n & \xleftarrow{\alpha_n^m} & F_m \\
 f \downarrow & & \downarrow h \\
 G & \xleftarrow{g} & H
 \end{array}$$

Note that the name fundamental sequence has not been standardized. Other names that have been used include generic sequence and Fraïssé sequence.

Theorem 1.4. *Let \mathcal{F} be a projective Fraïssé class with a fixed family of epimorphisms among the structures of \mathcal{F} . There exists a unique topological graph \mathbb{F} such that*

- (1) *for each $A \in \mathcal{F}$, there exists an epimorphism from \mathbb{F} to A ;*
- (2) *for $A, B \in \mathcal{F}$ and epimorphisms $f: \mathbb{F} \rightarrow A$ and $g: B \rightarrow A$ there exists an epimorphism $h: \mathbb{F} \rightarrow B$ such that $f = g \circ h$.*
- (3) *For every $\varepsilon > 0$ there is a graph $G \in \mathcal{F}$ and an epimorphism $f: \mathbb{F} \rightarrow G$ such that $\text{diam}(f^{-1}(x)) < \varepsilon$ for each x in $V(G)$.*

This topological graph \mathbb{F} is called the projective Fraïssé limit of \mathcal{F} .

Proof. The uniqueness of \mathbb{F} along with the first two conditions are precisely [11, Theorem 3.1]. For the third condition note that in the proof of [11, Theorem 3.1] it is shown that the projective Fraïssé limit \mathbb{F} is the inverse limit of a fundamental sequence of \mathcal{F} . Condition (3) then follows from the definition of a metric on the inverse limit space. \square

Definition 1.5. Given a topological graph K , if $E(K)$ is also transitive then it is an equivalence relation and K is known as a *prespace*. The quotient space $K/E(K)$ is called the *topological realization* of K and is denoted by $|K|$.

Theorem 1.6. *Each compact metric space is a topological realization of a topological graph.*

Proof. Let X be a compact metric space and let $f: C \rightarrow X$ be a surjective mapping from the Cantor set C . Define a topological graph K by putting $V(K) = C$ and $\langle a, b \rangle \in E(K)$ if and only if $f(a) = f(b)$. Then K is a compact topological graph, $E(K)$ is transitive, and $|K|$ is homeomorphic to X . \square

2. CONNECTEDNESS PROPERTIES OF TOPOLOGICAL GRAPHS

In this section we propose definitions of some connectedness properties of topological graphs analogous to respective definitions for continua. We have decided to keep the terminology original to continuum theory. Let us start with the definitions of connected and locally connected topological graphs as in [11].

Definition 2.1. Given a topological graph G , a subset S of $V(G)$ is *disconnected* if there are two nonempty closed subsets P and Q of S such that $P \cup Q = S$ and if $a \in P$ and $b \in Q$, then $\langle a, b \rangle \notin E(G)$. A

subset S of $V(G)$ is *connected* if it is not disconnected. A graph G is connected if $V(G)$ is connected.

Definition 2.2. A topological graph G is *locally connected* if it admits a basis of its topology consisting of connected sets in the above sense.

Theorem 2.3. *If $f: G \rightarrow H$ is an epimorphism between topological graphs and G is connected, then H is connected.*

Proof. Suppose H is disconnected. Then there are two disjoint nonempty closed subsets A and B of $V(H)$ such that $V(H) = A \cup B$ and there is no edges between A and B . So $V(G) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Since G is connected, there are vertices $a \in f^{-1}(A)$ and $b \in f^{-1}(B)$ such that $\langle a, b \rangle \in E(G)$, and thus $\langle f(a), f(b) \rangle \in E(H)$, a contradiction. \square

Definition 2.4. Given a graph G , a subset S of $V(G)$, and a vertex $a \in S$ the *component of S containing a* is the largest connected subset C of S that contains a ; in other words $C = \bigcup \{P \subseteq S : a \in P \text{ and } P \text{ is connected}\}$.

Definition 2.5. A topological graph G is called *hereditarily unicoherent* if for every two closed connected subgraphs P and Q of G the intersection $P \cap Q$ is connected. The graph G is *unicoherent* if in addition $P \cup Q = G$.

Notice that if a finite graph is unicoherent then it is hereditarily unicoherent.

Notation 2.6. If G is a topological graph and $S \subseteq V(G)$ by $G \setminus S$ we mean a topological graph F such that $V(F) = V(G) \setminus S$ and if $e = \langle a, b \rangle \in E(G)$ then $e \in F$ if and only if $\{a, b\} \subseteq V(F)$

The following theorem is known in continuum theory. Here we give the analogous theorem for topological graphs together with the proof.

Theorem 2.7. *If G is a hereditarily unicoherent topological graph and $\{G_\alpha : \alpha \in A\}$ is a family of connected subgraphs of G , then the intersection $\bigcap \{G_\alpha : \alpha \in A\}$ is connected.*

Proof. Suppose $J = \bigcap \{G_\alpha : \alpha \in A\}$ is disconnected. Then there are two disjoint closed subsets H and K of $V(J)$ such that $H \cap K = \emptyset$ and $H \cup K = V(J)$. Let H^* and K^* be disjoint open sets containing H and K respectively. The collection $\{G \setminus G_\alpha : \alpha \in A\} \cup (H^* \cup K^*)$ is an open cover of G . By compactness, there is a finite open subcover $G \setminus G_{\alpha_1}, \dots, G \setminus G_{\alpha_n}, H^* \cup K^*$ of G . Taking the complements we have that $H \cup K \subseteq G_{\alpha_1} \cap \dots \cap G_{\alpha_n} \subseteq H^* \cup K^*$ and $G_{\alpha_1} \cap \dots \cap G_{\alpha_n}$ is, by hereditary unicoherence, a connected graph, a contradiction.

□

Definition 2.8. We say that a topological graph G is an *arc* if it is connected and there are two vertices $a, b \in V(G)$ such that for every $x \in V(G) \setminus \{a, b\}$ the graph $G \setminus \{x\}$ is not connected. The vertices a and b are called end vertices of the arc and we say that G joins a and b .

In topology there is only one, up to homeomorphism, metric arc, while for topological graphs we have finite arcs, countable arcs as well as uncountable arcs.

Example 2.9. Let G be the topological graph whose set of vertices is the Cantor ternary set, and the edges are defined by $\langle a, b \rangle \in E(G)$ if and only if $a = b$ or a and b are end vertices of the same deleted interval. Then G is an uncountable arc, whose topological realization is homeomorphic to $[0, 1]$.

Definition 2.10. A topological graph G is called *arcwise connected* if for every two vertices $a, b \in V(G)$ there is a subgraph of G that is an arc and contains a and b .

Definition 2.11. A hereditarily unicoherent and arcwise connected finite graph is called a *dendroid*. A locally connected dendroid is called a *dendrite*.

Theorem 2.12. *For a connected finite graph G the following conditions are equivalent:*

- (1) G is a tree;
- (2) G is unicoherent;
- (3) G is a dendroid;
- (4) G is a dendrite;

Proof. The implications (1) \implies (2) \implies (3) \implies (4) follow from definitions. To see (4) \implies (1) note that a finite dendrite cannot contain a cycle, so it is a tree. □

The following observations follow from the fact that a topological graph G is connected if and only if its topological realization $|G|$ is connected.

Observation 2.13. *If G is a hereditarily unicoherent topological graph and $E(G)$ is transitive, then its topological realization $|G|$ is a hereditarily unicoherent continuum.*

Observation 2.14. *If a topological graph G is an arc and $E(G)$ is transitive, then its topological realization $|G|$ is an arc or a point.*

Observation 2.15. *If a topological graph G is arcwise connected and $E(G)$ is transitive, then its topological realization $|G|$ is arcwise connected.*

Observation 2.16. *If a topological graph G is a dendroid and $E(G)$ is transitive, then its topological realization $|G|$ is a dendroid.*

Observation 2.17. *If a topological graph G is a dendrite and $E(G)$ is transitive, then its topological realization $|G|$ is a dendrite.*

The following Theorem provides a sufficient condition in order for a projective Fraïssé family to have transitive set of edges in the projective Fraïssé limit.

Theorem 2.18. *Suppose that \mathcal{G} is a projective Fraïssé family of graphs and for every $G \in \mathcal{G}$, for every $a, b, c \in V(G)$ such that $\langle a, b \rangle \in E(G)$ and $\langle b, c \rangle \in E(G)$ there is a graph H and an epimorphism $f_G^H: H \rightarrow G$ such that for every vertices $p, q, r \in V(H)$ such that $f_G^H(p) = a$, $f_G^H(q) = b$, and $f_G^H(r) = c$ we have $\langle p, q \rangle \notin E(H)$ or $\langle q, r \rangle \notin E(H)$. Then there are not distinct vertices a, b, c in the Fraïssé limit \mathbb{G} such that the edges $\langle a, b \rangle$ and $\langle b, c \rangle$ are in \mathbb{G} hence \mathcal{G} has a transitive set of edges.*

Proof. Let \mathcal{G} be a family that satisfies the assumptions of the Theorem. Suppose on the contrary that there are three vertices $a, b, c \in \mathbb{G}$ such that $\langle a, b \rangle, \langle b, c \rangle \in E(\mathbb{G})$. Let a graph $G \in \mathcal{G}$ and an epimorphism $f_G: \mathbb{G} \rightarrow G$ be such that $f_G(a), f_G(b)$, and $f_G(c)$ are three distinct vertices of G . Then $\langle f_G(a), f_G(b) \rangle, \langle f_G(b), f_G(c) \rangle \in E(G)$, and thus, by our assumption, there is a graph H and an epimorphism $f_G^H: H \rightarrow G$ such that for an epimorphism $f_H: \mathbb{G} \rightarrow H$ satisfying $f_G = f_G^H \circ f_H$ we have $\langle f_H(a), f_H(b) \rangle \notin E(H)$ or $\langle f_H(b), f_H(c) \rangle \notin E(H)$. This contradicts the fact that f_H maps edges to edges. \square

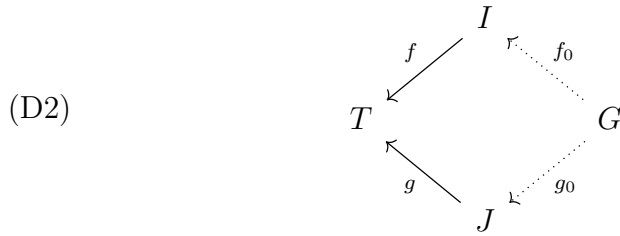
Theorem 2.19. *If \mathcal{T} is a projective Fraïssé family of trees, and \mathbb{T} is a projective Fraïssé limit of \mathcal{T} , then \mathbb{T} is hereditarily unicoherent.*

Proof. Suppose \mathbb{T} is not hereditarily unicoherent. Then there exist closed connected subsets P and Q of $V(\mathbb{T})$ such that $P \cap Q$ is not connected. Let $p \in P \setminus Q$ and $q \in Q \setminus P$. By Condition 3 of Theorem 1.4, choosing ε small enough, there exists a tree G and a epimorphism $f_G: \mathbb{T} \rightarrow G$ such that $f_G(p) \notin f_G(Q)$, $f_G(q) \notin f_G(P)$, and $f_G(P) \cap f_G(Q)$ is not connected. This contradicts the fact that G is a tree. \square

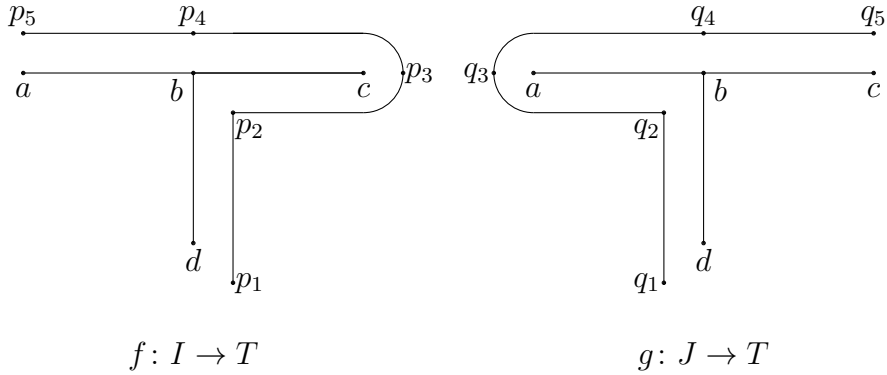
3. MONOTONE EPIMORPHISMS

First, we want to show an example that usually epimorphisms between trees do not have amalgamations that are connected graphs. That is why we need to restrict considered classes of epimorphisms.

Example 3.1. There exist a triod T , arcs I and J for which there is no connected graph G and epimorphisms f_0 and g_0 such that the diagram below commutes.



The triod T has the center b and the end vertices a , c , and d ; the arc I has vertices p_1, p_2, p_3, p_4 , and p_5 , similarly the arc J has vertices q_1, q_2, q_3, q_4 , and q_5 . The epimorphisms f and g are pictured below. Precisely, we have $f(p_1) = d$, $f(p_2) = b$, $f(p_3) = c$, $f(p_4) = b$, and $f(p_5) = a$; similarly, $g(q_1) = d$, $g(q_2) = b$, $g(q_3) = a$, $g(q_4) = b$, and $g(q_5) = c$.



Suppose that there is a connected graph G and epimorphisms $f_0: G \rightarrow I$ and $g_0: G \rightarrow J$ such that the diagram (D2) commutes. Let $x_0 \in (f_0)^{-1}(p_1)$ and let x_0, x_1, \dots, x_n be a sequence of vertices of G such that:

- (1) for each $i \in \{0, 1, \dots, n-1\}$ we have $\langle x_i, x_{i+1} \rangle \in E(G)$;
- (2) $x_0, x_1, \dots, x_{n-1} \in (f \circ f_0)^{-1}(\{b, d\})$;
- (3) $x_n \notin (f \circ f_0)^{-1}(\{b, d\})$.

Then we have

- (1) $f_0(x_0) = p_1$,
- (2) $f_0(x_0), f_0(x_1), \dots, f_0(x_{n-1}) \in \{p_1, p_2\}$;
- (3) $f_0(x_n) = p_3$;
- (4) $g_0(x_0) = q_1$,
- (5) $g_0(x_0), g_0(x_1), \dots, g_0(x_{n-1}) \in \{q_1, q_2\}$;
- (6) $g_0(x_n) = q_3$.

By conditions (3) and (6) we have that $f(f_0(x_n)) = c$, while $g(g_0(x_n)) = a$, so the diagram (D2) does not commute, a contradiction.

In this section we consider the category $\mathcal{T}_{\mathcal{M}}$ of trees with morphisms being monotone epimorphisms. We start with necessary definitions.

Definition 3.2. Given two topological graphs G and H an epimorphism $f: G \rightarrow H$ is called *monotone* if the preimage of a connected subset of $V(H)$ is a connected subset of $V(G)$, or, equivalently (see [7, Lemma 1.1]) if the preimage of every vertex in $V(H)$ is connected.

The following example shows that the concept of monotone epimorphism for graphs is not exactly how continuum theory sees monotone maps between continua.

Example 3.3. There is a monotone epimorphism from a cyclic graph onto an arc.

Let G be a complete graph with three vertices a, b , and c , i.e. a graph where $E(G) = V(G)^2$, and let H be an arc with two vertices p and q , and $E(H) = V(H)^2$. Define $f: G \rightarrow H$ by $f(a) = p$ and $f(b) = f(c) = q$. The reader can verify that f is a monotone epimorphism.

The following theorem is well known in continuum theory but we include it in this new setting of topological graphs.

Theorem 3.4. *If $f: G \rightarrow H$ is a monotone epimorphism between topological graphs and G is an arc, then H is an arc and the images of end vertices of G are end vertices of H .*

Proof. Denote the end vertices of G by a and b . We need to show that every vertex in $V(H) \setminus \{f(a), f(b)\}$ disconnects H . Let $y \in V(H) \setminus \{f(a), f(b)\}$; then, since G is an arc the graph $G \setminus f^{-1}(y)$ is disconnected. Let $G \setminus f^{-1}(y)$ be the union of two disjoint graphs $G \setminus f^{-1}(y) = U \cup V$. Thus $V(H) \setminus \{y\} = f(U) \cup f(V)$, so it is disconnected as needed. \square

Theorem 3.5. *If $f: G \rightarrow H$ is a monotone epimorphism between finite graphs and G is a tree, then H is a tree and the images of end vertices of G are end vertices of H .*

Proof. Suppose on the contrary that there are two vertices $f(a)$ and $f(b)$ in H and two different arcs I and J joining $f(a)$ and $f(b)$. Since $f^{-1}(I)$ and $f^{-1}(J)$ are connected, there are arcs $A \subseteq f^{-1}(I)$ and $B \subseteq f^{-1}(J)$ joining the vertices a and b . Then $f(A) = I$ and $f(B) = J$, so A and B are two different arcs joining a and b in G , contrary to the assumption that G is a tree. \square

Theorem 3.6. *If $f: G \rightarrow H$ is a monotone epimorphism between trees and K is a subtree of G , then $f|_K: K \rightarrow f(K)$ is a monotone epimorphism.*

Proof. Let $y \in V(f(K))$; we need to show that $(f|_K)^{-1}(y)$ is connected. It is enough to observe that $(f|_K)^{-1}(y) = f^{-1}(y) \cap K$, so it is connected as the intersection of subtrees of G . \square

Theorem 3.7. *If \mathcal{G} is a projective Fraïssé family of graphs with monotone epimorphisms and \mathbb{G} is a projective Fraïssé limit of \mathcal{G} , then for every $G \in \mathcal{G}$ any epimorphism $f_G: \mathbb{G} \rightarrow G$ is monotone.*

Proof. Suppose the contrary; then there is a graph $G \in \mathcal{G}$, a vertex $a \in G$ and an epimorphism $f_G: \mathbb{G} \rightarrow G$ such that $f_G^{-1}(a)$ is the disjoint union of two nonempty closed subsets A and B . Choose $\varepsilon > 0$ such that $d(a, b) > \varepsilon$ for every $a \in A$ and $b \in B$. By conditions (2) and (3) of Theorem 1.4 and amalgamation we may obtain a graph $H \in \mathcal{G}$, an epimorphism $f_G^H: H \rightarrow G$, and an epimorphism $f_H: \mathbb{G} \rightarrow H$ be such that $f_G = f_G^H \circ f_H$ and f_H is an ε epimorphism. Then $f_H(A)$ and $f_H(B)$ are two disjoint nonempty subsets of H and $(f_G^H)^{-1}(a) = f_H(A) \cup f_H(B)$ contrary to monotonicity of f_G^H . \square

Theorem 3.8. *If \mathcal{T} is a projective Fraïssé family of finite arcs with monotone epimorphisms, and \mathbb{T} is a projective Fraïssé limit of \mathcal{T} , then \mathbb{T} is an arc.*

Proof. First we claim there are two vertices a and b in \mathbb{T} such that for any $T \in \mathcal{T}$ and any monotone epimorphism $f_T: \mathbb{T} \rightarrow T$, $f_T(a)$ and $f_T(b)$ are end vertices of T . Let $\{T_n, \alpha_n\}$ be a fundamental sequence for \mathcal{T} . The monotone epimorphisms α_n take end vertices to end vertices so there are exactly two vertices $a = (a_n)$ and $b = (b_n)$ in \mathbb{T} such that for each n , a_n and b_n is an end vertex for T_n . For $T \in \mathcal{T}$ and $f_T: \mathbb{T} \rightarrow T$ there is an n and monotone epimorphisms $g: T_n \rightarrow T$ and $f_{T_n}: \mathbb{T} \rightarrow T_n$ such that $f_T = g \circ f_{T_n}$. Thus f_T maps the vertices a and b to end vertices of T as claimed.

We need to show that any vertex $x \in \mathbb{T} \setminus \{a, b\}$ disconnects \mathbb{T} . For an arc $T \in \mathcal{T}$ and a given monotone epimorphism $f_T: \mathbb{T} \rightarrow T$ define U_{f_T} and V_{f_T} as components of $T \setminus \{f_T(x)\}$ containing the vertices $f_T(a)$ and $f_T(b)$ respectively. Note that for $T, S \in \mathcal{T}$ and a

monotone epimorphism $f_S^T: T \rightarrow S$ such that $f_S = f_S^T \circ f_T$ we have $(f_S^T)^{-1}(U_{f_S}) \subseteq U_{f_T}$, and thus $f_S^{-1}(U_{f_S}) \subseteq f_T^{-1}(U_{f_T})$. Letting $U = \bigcup \{U_{f_T} : f_T \text{ is an epimorphism between trees in } \mathcal{T}\}$ and $V = \bigcup \{V_{f_T} : f_T \text{ is an epimorphism between trees in } \mathcal{T}\}$ we have $\mathbb{T} \setminus \{x\} = U \cup V$ and $U \cap V = \emptyset$, so $\mathbb{T} \setminus \{x\}$ is not connected as required. \square

Theorem 3.9. *If \mathcal{T} is a projective Fraïssé family of finite trees with monotone epimorphisms, and \mathbb{T} is a projective Fraïssé limit of \mathcal{T} , then \mathbb{T} is arcwise connected.*

Proof. Let a, b be two vertices of \mathbb{T} . We will construct an arc joining a and b . Let us use notations as is the proof of Theorem 3.8: that is, for trees $T, T_1, T_2 \in \mathcal{T}$, f_T is a monotone epimorphism that maps the limit \mathbb{T} onto the tree T , while $f_{T_2}^{T_1}$ is a monotone epimorphism between trees in \mathcal{T} . For a given $T \in \mathcal{T}$, let J_T be the arc between $f_T(a)$ and $f_T(b)$ and observe that $f_{T_2}^{T_1}(J_{T_1}) \subseteq J_{T_2}$, and thus $\{J_T : T \in \mathcal{T}\}$ is a projective Fraïssé family, whose limit \mathbb{J} is, by Theorem 3.8, an arc joining a and b . \square

Corollary 3.10. *If \mathcal{T} is a projective Fraïssé family of trees with monotone epimorphisms, and \mathbb{T} is a projective Fraïssé limit of \mathcal{T} , then \mathbb{T} is a dendrite.*

Proof. The limit \mathbb{T} is arcwise connected by Theorem 3.9, it is hereditarily unicoherent by Theorem 2.19, and it is locally connected by [11, Theorem 2.1]. \square

Theorem 3.11. *The category $\mathcal{T}_{\mathcal{M}}$ is a projective Fraïssé family.*

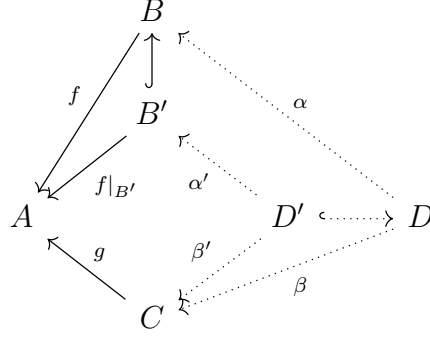
Proof. Conditions (1) and (2) are obvious.

For condition (4) we will use induction on the sum of the number of edges in B and C . If B and C each have just one degenerate edge, that is, B and C are singletons, then A has only one vertex and we can take D to be just a single vertex.

Now assume there is a natural number N and such that for any trees A, B , and C with the sum of the number of edges in B and C is less than or equal to N and $f: B \rightarrow A$ and $g: C \rightarrow A$ are monotone epimorphisms then there exists a tree D and monotone epimorphisms $\alpha: D \rightarrow B$ and $\beta: D \rightarrow C$ such that $f \circ \alpha = g \circ \beta$.

Suppose A, B and C are trees such that the sum of the number of edges in B and C is $N + 1$ and there are monotone epimorphisms $f: B \rightarrow A$ and $g: C \rightarrow A$. Let $\langle a, b \rangle$ be an edge in B such that b is an end vertex of B . Let B' be the finite subtree of B without the edge $\langle a, b \rangle$ (still containing the vertex a). There are two possibilities, $f|_{B'}$ is surjective or it is not.

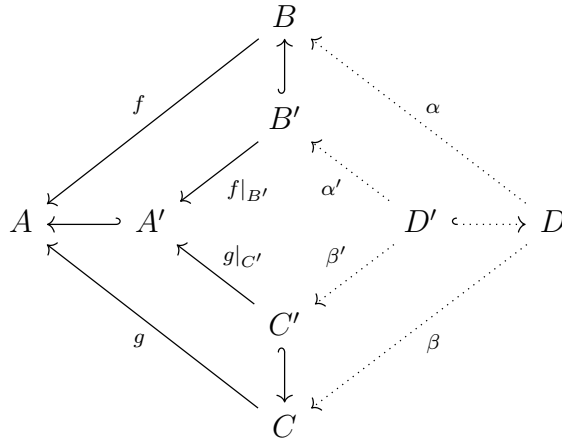
Consider the case that $f|_{B'}$ is surjective. By the inductive hypothesis there exist a tree D' and monotone epimorphisms $\alpha': D' \rightarrow B'$ and $\beta': D' \rightarrow C$ such that $f|_{B'} \circ \alpha' = g \circ \beta'$ (see the diagram below).



Note that $f(b) = f(a)$ otherwise if $f(b) \neq f(a)$ then there exist $c \in B'$ such that $f(c) = f(b)$. But then $\{b, c\} \in f^{-1}(f(b))$ and $a \notin f^{-1}(f(b))$ so $f^{-1}(f(b))$ is not connected, contradicting f being monotone.

Chose $c \in (\alpha')^{-1}(a)$ and let $D = D' \cup \langle c, d \rangle$ where d is a vertex not in D' . Let $\alpha: D \rightarrow B$ be the extension of α' with $\alpha(d) = b$ and β be the extension of β' with $\beta(d) = \beta'(c)$. Then D is the desired amalgamation.

Now consider the case that $f|_{B'} = A'$, a proper subtree of A . So $E(A) = E(A') \cup \{\langle f(a), f(b) \rangle\}$. Let $C' = g^{-1}(A')$ and $F = g^{-1}(\{f(b)\})$. Then A' , B' and C' together with $f|_{B'}: B' \rightarrow A'$ and $g|_{C'}: C' \rightarrow A'$ satisfy the inductive hypothesis. So there exist a tree D' and monotone epimorphisms $\alpha': D' \rightarrow B'$ and $\beta': D' \rightarrow C'$ such that $f|_{B'} \circ \alpha' = g|_{C'} \circ \beta'$ (see the diagram below).



Let $\langle p, q \rangle$ be an edge in C with $p \in g^{-1}(f(a))$ and $q \in F$. Next let F' be an isomorphic copy of F and $D = D' \cup F' \cup \langle p', q' \rangle$ where

$p' \in \beta'^{-1}(p)$ and q' is the image of q under the isomorphism. If α is the extension of α' with $\alpha(c) = b$ for all $c \in F'$ and β is the extension of β' such that for all $x \in F'$, $\beta(x)$ is the image of x under the isomorphism between F and F' , then the tree D together with α and β is the desired amalgamation.

Finally, to prove condition (3) we may let A be the tree consisting of a single vertex and f and g be constant maps, then the tree D and maps α and β from the previous part satisfy condition (3). \square

Recall (see [4, (6), p. 490]) that there is only one, up to homeomorphisms, dendrite, called D_3 , that satisfies the following conditions:

- (1) each ramification point is of order 3;
- (2) the set of ramification points is dense.

We want to show that the topological realization of the projective Fraïssé limit of the category $\mathcal{T}_{\mathcal{M}}$ is homeomorphic to the dendrite D_3 . We divide the proof into two steps.

Theorem 3.12. *The projective Fraïssé limit of $\mathcal{T}_{\mathcal{M}}$ has transitive set of edges and the topological realization of the Fraïssé limit of $\mathcal{T}_{\mathcal{M}}$ is a dendrite with a dense set of ramification points.*

Proof. First we show that the family $\mathcal{T}_{\mathcal{M}}$ satisfies the hypothesis of Theorem 2.18 and thus has a transitive set of edges. Let $G \in \mathcal{T}_{\mathcal{C}}$ and $a, b, c \in V(G)$ such that $\langle a, b \rangle, \langle b, c \rangle \in E(G)$. Define $H \in \mathcal{T}_{\mathcal{C}}$ by $V(H) = V(G) \cup \{a', b'\}$, $E(H) = E(G) \cup \{\langle a, a' \rangle, \langle a', b' \rangle, \langle b', b \rangle\} \setminus \{\langle a, b \rangle\}$, and the epimorphism f_G^H where $f_G^H(a') = a$, $f_G^H(b') = b$ and f_G^H is the identity otherwise. Then, for $p \in \{a, a'\}$, $q \in \{b, b'\}$, and $r = c$ we have either $\langle p, q \rangle \notin E(H)$ or $\langle q, r \rangle \notin E(H)$.

The projective Fraïssé limit \mathbb{D} of $\mathcal{T}_{\mathcal{M}}$ is a dendrite by Corollary 3.10, so its topological realization $|\mathbb{D}|$ is a (topological) dendrite by Corollary 2.17. It remains to show that the set of ramification points of $|\mathbb{D}|$ is dense.

Let us introduce the necessary notation. Let $\varphi: \mathbb{D} \rightarrow |\mathbb{D}|$ be the quotient mapping. To prove the density of the set of ramification points of $|\mathbb{D}|$ suppose on the contrary that U is an open connected subset of $|\mathbb{D}|$ that contains no ramification point, i.e U is homeomorphic to $(0, 1)$. Then $\varphi^{-1}(U)$ is an open subset of \mathbb{D} . By Property (3) of Theorem 1.4, there is a tree G , a monotone epimorphism $f_G: \mathbb{D} \rightarrow G$, and $a \in V(G)$ such that $f_G^{-1}(a) \subseteq \varphi^{-1}(U)$. Define a graph H by putting $V(H) = V(G) \cup \{b, c, d\}$ and $E(H) = E(G) \cup \{\langle a, b \rangle, \langle b, c \rangle, \langle b, d \rangle\}$.

Finally, define the monotone epimorphism $f_G^H: H \rightarrow G$ by

$$f_G^H(x) = \begin{cases} x & \text{if } x \in G \\ a & \text{if } x \in \{b, c, d\}. \end{cases}$$

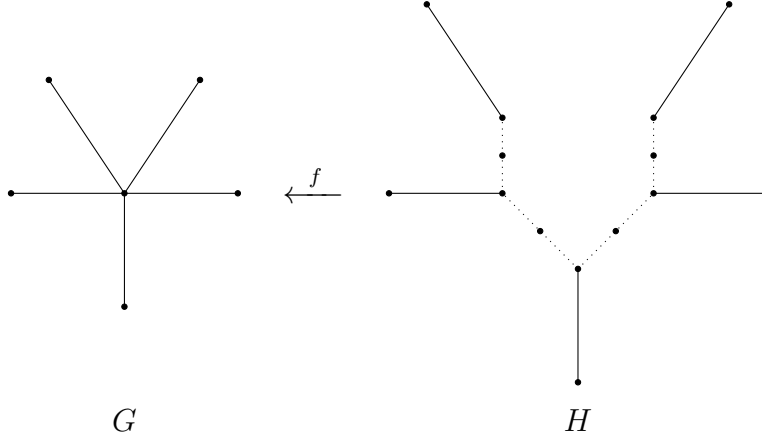
Let $f_H: \mathbb{D} \rightarrow H$ be an epimorphism such that $f_G = f_G^H \circ f_H$; note that by Theorem 3.7 the epimorphisms f_G and f_H are monotone. Thus, the three sets $A = \varphi(f_H^{-1}(\{a, b\}))$, $B = \varphi(f_H^{-1}(\{b, c\}))$, and $C = \varphi(f_H^{-1}(\{b, d\}))$ are continua that satisfy $A \cup B \cup C \subseteq U$, $A \not\subseteq B \cup C$, $B \not\subseteq A \cup C$, $C \not\subseteq A \cup B$, and $A \cap B \cap C \neq \emptyset$. This contradicts the fact that U is homeomorphic to $(0, 1)$. □

Definition 3.13. Denote by $\mathcal{T}_{\mathcal{M}_3}$ the subfamily of $\mathcal{T}_{\mathcal{M}}$ such that each member of $\mathcal{T}_{\mathcal{M}_3}$ satisfies:

- each vertex is of order at most 3;
- no two vertices of order 3 are connected by an edge.

Proposition 3.14. *The family $\mathcal{T}_{\mathcal{M}_3}$ is cofinal in $\mathcal{T}_{\mathcal{M}}$ and thus the projective Fraïssé limits of $\mathcal{T}_{\mathcal{M}_3}$ and $\mathcal{T}_{\mathcal{M}}$ are isomorphic.*

Proof. It is enough to construct, for any graph G in $\mathcal{T}_{\mathcal{M}}$, a graph H in $\mathcal{T}_{\mathcal{M}_3}$ and a monotone epimorphism $f: H \rightarrow G$. The idea is pictured below: we replace each ramification vertex with a set of vertices of orders 2 and 3 as pictured. The epimorphism f shrinks the subgraph of H pictured by dotted lines to the central vertex of G . The details are left to the reader.



□

Theorem 3.15. *The topological realization of the projective Fraïssé limit of $\mathcal{T}_{\mathcal{M}}$ is homeomorphic to the dendrite D_3 .*

Proof. By Theorem 3.12 the topological realization $|\mathbb{D}|$ of the projective Fraïssé limit of $\mathcal{T}_{\mathcal{M}}$ is a dendrite with a dense set of ramification points; it remains to show that every ramification point of $|\mathbb{D}|$ has order three. Suppose on the contrary that $p \in |\mathbb{D}|$ is a point of order four or more. Then there are four arcs A, B, C, D in $|\mathbb{D}|$ such that the intersection any two of them is $\{p\}$.

Note that the transitivity of the edge relation on \mathbb{D} implies that $\varphi^{-1}(p)$ is a complete graph; on the other hand, by hereditary unicoherence of \mathbb{D} the set $\varphi^{-1}(p)$ has at most two points. Denote $\varphi^{-1}(p) = \{p_1, p_2\}$ with a possibility that $p_1 = p_2$ and note that the sets $\varphi^{-1}(A)$, $\varphi^{-1}(B)$, $\varphi^{-1}(C)$, and $\varphi^{-1}(D)$ contains arcs $\alpha, \beta, \gamma, \delta$ in \mathbb{D} and that the intersection of any two of these arcs is $\{p_1, p_2\}$.

Let G be a tree and $f_G: \mathbb{D} \rightarrow G$ be a monotone epimorphism such that:

- (1) $f_G(\alpha)$, $f_G(\beta)$, $f_G(\gamma)$, and $f_G(\delta)$ are nondegenerate subsets of $V(G)$;
- (2) $G \in \mathcal{T}_{\mathcal{M}3}$;

By Theorem 3.4 the sets $f_G(\alpha)$, $f_G(\beta)$, $f_G(\gamma)$, and $f_G(\delta)$ are arcs. We claim that the intersection of any two of them is $\{f_G(p_1), f_G(p_2)\}$. Suppose on the contrary that $r \in f_G(\alpha) \cap f_G(\beta) \setminus \{f_G(p_1), f_G(p_2)\}$; then $f_G^{-1}(r)$ is a connected subset of $V(\mathbb{D})$ that intersects both $\alpha \setminus \{p_1, p_2\}$ and $\beta \setminus \{p_1, p_2\}$ contrary to hereditary unicoherence of \mathbb{D} . The claim contradicts the condition that both $f_G(p_1)$ and $f_G(p_2)$ are points of order at most 3 and not both of them are of order 3. \square

4. CONFLUENT EPIMORPHISMS

Motivated by the definition of a confluent map between continua (see [3]) we give an analogous definition for graphs.

Definition 4.1. Given two topological graphs G and H an epimorphism $f: G \rightarrow H$ is called *confluent* if for every connected subset Q of $V(H)$ and every component C of $f^{-1}(Q)$ we have $f(C) = Q$. Equivalently, if for every connected subset Q of $V(H)$ and every vertex $a \in V(G)$ such that $f(a) \in Q$ there is a connected set C of $V(G)$ such that $a \in C$ and $f(C) = Q$. Clearly, every monotone epimorphism is confluent.

Observation 4.2. *If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are confluent epimorphisms between topological graphs, then $g \circ f: X \rightarrow Z$ is a confluent epimorphism.*

Theorem 4.3. *Given two finite graphs G and H the following conditions are equivalent for an epimorphism $f: G \rightarrow H$:*

- (1) f is confluent;
- (2) for every edge $P \in E(H)$ and for every vertex $a \in V(G)$ such that $f(a) \in P$, there is an edge $E \in E(G)$ and a connected set $R \subseteq V(G)$ such that $E \cap R \neq \emptyset$, $f(E) = P$, $a \in R$, and $f(R) = \{f(a)\}$.
- (3) for every edge $P \in E(H)$ and every component C of $f^{-1}(P)$ there is an edge E in C such that $f(E) = P$.

Proof. (1) \Rightarrow (2): Let $P \in E(H)$ and $a \in V(G)$ such that $f(a) \in P$. Then there exists a connected subset $C \subseteq V(G)$ such that $a \in C$ and $f(C) = P$. Since C is connected there is an edge $E \in C$ such that $f(E) = P$. If $a \in E$ let $R = \{a\}$ otherwise let R be the component of $f^{-1}(f(a))$ containing a . In either case (2) is satisfied.

(2) \Rightarrow (3): This is immediate.

(3) \Rightarrow (1): Given a connected set Q in H start with an edge in Q and use condition (3) edge-by-edge to build a connected set containing one of the end vertices of the initial edge that maps onto Q .

□

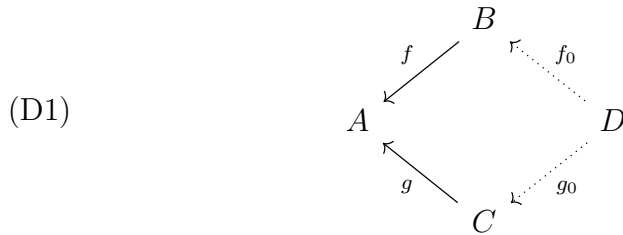
Theorem 4.4. *If \mathcal{F} is a projective Fraïssé family of graphs with confluent epimorphisms and \mathbb{F} is a projective Fraïssé limit of \mathcal{F} , then for every graph $G \in \mathcal{F}$ every epimorphism $f_G: \mathbb{F} \rightarrow G$ is confluent.*

Proof. Let $\{F_n, \alpha_n\}$ be a fundamental sequence for \mathcal{F} . It is easy to see that the proof of [9, Theorem 2.1.23] can be adapted to the present language to show that the canonical projections $\alpha_m^\infty: \varprojlim \{F_n, \alpha_n\} \rightarrow F_m$ are confluent. Because $\{F_n, \alpha_n\}$ is a fundamental sequence there is an m and a confluent epimorphism $f: F_m \rightarrow G$. Then $f_G = f \circ \alpha_m$ is confluent.

□

Definition 4.5. An epimorphism $f: G \rightarrow H$ between graphs is called *light* if for any vertex $h \in V(H)$ and for any $a, b \in f^{-1}(h)$ we have $\langle a, b \rangle \notin E(G)$.

Definition 4.6. Recall the diagram (D1).



For given graphs A, B, C and epimorphisms $f: B \rightarrow A$, $g: C \rightarrow A$ by *standard amalgamation procedure* we mean the graph D defined by $V(D) = \{(b, c) \in B \times C : f(b) = g(c)\}$; $E(D) = \{\langle (b, c), (b', c') \rangle : \langle b, b' \rangle \in E(B) \text{ and } \langle c, c' \rangle \in E(C)\}$; and $f_0((b, c)) = b$, $g_0((b, c)) = c$. Note that f_0 and g_0 are epimorphisms and $f \circ f_0 = g \circ g_0$.

Proposition 4.7. *Using the standard amalgamation procedure and the notation of diagram (D1), if the epimorphism f is light, then the epimorphism g_0 is light.*

Proof. Suppose g_0 is not light. Then there is a vertex $h \in V(C)$ such that $(a_1, h), (a_2, h) \in g_0^{-1}(h)$, and $\langle (a_1, h), (a_2, h) \rangle \in E(D)$. Then, $\langle a_1, a_2 \rangle \in E(B)$ and $a_1, a_2 \in f^{-1}(g(h))$, contradicting f being light. \square

Proposition 4.8. *Using the standard amalgamation procedure and the notation of the diagram (D1), if the epimorphism f is monotone, then the epimorphism g_0 is monotone.*

Proof. Fix a vertex $c \in V(C)$ and consider $g_0^{-1}(c) = \{\langle x, c \rangle : f(x) = g(c)\}$. This set is isomorphic to $f^{-1}(g(c))$ which is connected by monotonicity of f . \square

Proposition 4.9. *Using the standard amalgamation procedure and the notation of diagram (D1), if the epimorphism f is confluent, then the epimorphism g_0 is confluent.*

Proof. Let $\langle c, c' \rangle \in E(C)$ and $(b, c) \in V(D)$. By Theorem 4.3 we need to find an edge $\langle d_1, d_2 \rangle \in D$ that is mapped by g_0 onto $\langle c, c' \rangle$ and a connected set R such that $(b, c), d_1 \in R$, and $g_0(R) = \{c\}$. Since f is confluent, there are vertices $b_1, b_2 \in B$ and a connected set R_B such that $\langle b_1, b_2 \rangle \in E(B)$, $b, b_1 \in R_B$ and $f(\langle b_1, b_2 \rangle) = \langle g(c), g(c') \rangle$. It is enough to put $d_1 = (b_1, c)$, $d_2 = (b_2, c')$, and $R = R_B \times \{c\}$. \square

Corollary 4.10. *Using the standard amalgamation procedure and the notation of diagram (D1), if the graphs A, B, C are connected and the epimorphisms f and g are confluent, then for each component P of the graph D we have $f_0(P) = B$ and $g_0(P) = C$.*

Corollary 4.11. *Using the standard amalgamation procedure and the notation of diagram (D1), if the graphs A, B, C are connected and the epimorphisms f and g are confluent, then there is a connected graph D and confluent epimorphisms f_0 and g_0 that make the diagram commutative.*

Proof. This follows from Corollary 4.10 by taking one of the components as the graph D . \square

Proposition 4.12. *Given an epimorphism $f: G \rightarrow H$ between graphs there is a graph M and epimorphisms $m: G \rightarrow M$, $l: M \rightarrow H$ such that $f = l \circ m$, m is monotone, and l is light.*

Proof. Define an equivalence relation \sim on $V(G)$ by $x \sim y$ if and only if $f(x) = f(y)$ and x and y are in the same component of $f^{-1}(f(x))$. Putting $V(M) = V(G)/\sim$, $m: V(G) \rightarrow V(M)$ as the projection, $\langle [x]_{\sim}, [y]_{\sim} \rangle \in E(M)$ if and only if $\langle x, y \rangle \in E(G)$, and $l([x]_{\sim}) = f(x)$ one can verify that the conclusions of the Proposition are satisfied. \square

Proposition 4.13. *Consider the epimorphisms $f: F \rightarrow G$ and $g: G \rightarrow H$ between topological graphs. If the composition $g \circ f: F \rightarrow H$ is confluent, then g is confluent.*

Proof. Let E be a closed connected subset of H , and let $a \in G$ be a vertex such that $g(a) \in E$. We need to find a closed connected subset S of G that contains a and such that $g(S) = E$. Since $g \circ f$ is confluent, there is a closed connected subset C of F that contains a vertex in $f^{-1}(a)$ and such that $(g \circ f)(C) = E$. Then $S = f(C)$ satisfies the requirements. \square

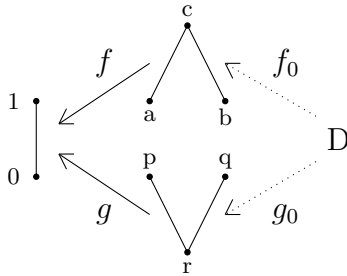
Corollary 4.14. *Under the hypothesis of Proposition 4.12 if the epimorphism f is confluent, then both m and l are confluent.*

Proof. The epimorphism m is confluent since it is monotone, while l is confluent by Proposition 4.13. \square

5. ROOTED TREES

The following example shows that confluent epimorphisms on trees does not behave well.

Example 5.1. Let A consist of just one edge $\langle 0, 1 \rangle$, B be an arc with three vertices $\{a, b, c\}$ and two edges $\langle a, c \rangle$, $\langle c, b \rangle$, and C be again an arc with three vertices $\{p, q, r\}$ and two edges $\langle p, r \rangle$, $\langle r, q \rangle$. Let $f: B \rightarrow A$ be such that $f(a) = f(b) = 0$ and $f(c) = 1$ and let $g: C \rightarrow A$ be such that $g(p) = g(q) = 1$ and $g(r) = 0$, see figure below. It is easy to see that f and g are confluent epimorphisms.



Note that the standard amalgamation procedure gives the graph D to be the cycle $(c, p), (a, r), (c, q), (b, r), (c, p)$ hence not a tree.

In fact, no amalgamation of the trees A, B , and C yields a tree. To see this assume there is an amalgamation of A, B , and C , that is, there is finite tree D and confluent epimorphisms $f_0: D \rightarrow B$ and $g_0: D \rightarrow C$ such that $g \circ g_0 = f \circ f_0$.

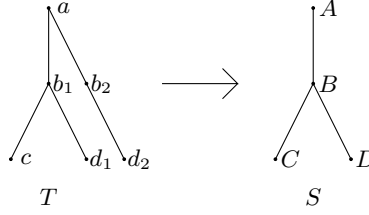
We will show there must then exist an infinite sequence of distinct vertices in the finite tree D . To this end, let $\langle a_1, p_1 \rangle$ be an edge in D such that f_0 maps onto the edge $\langle a, c \rangle$ in B with $f_0(a_1) = a$, $f_0(p_1) = c$ and $g_0(p_1) = p$. We can select a vertex p_1 that satisfies the last equality since $f(c) = 1$. Next let $\langle p_1, b_1 \rangle$ be an edge in D that g_0 maps onto the edge $\langle p, r \rangle$ in C . So $g_0(b_1) = r$ and we may select b_1 so that $f_0(b_1) = b$. Note that $a_1 \neq b_1$. In the same manner we can select a vertex q_1 in D such that f_0 maps the edge $\langle b_1, q_1 \rangle$ onto the edge $\langle a, c \rangle$ in B with $g_0(q_1) = q$ and $q_1 \neq p_1$. Inductively we obtain sequences of vertices $(a_n), (b_n), (p_n)$, and (q_n) where unicoherence of the tree D guarantees that no vertices in a sequence equals a vertices earlier in the sequence. Since D is a finite tree we have a contradiction to our assumption that there was an amalgamation of A, B , and C .

Because of Example 5.1 we will consider rooted trees and whenever we have an epimorphism between rooted trees we will assume that the epimorphism preserves orders defined by the selected points. Precise definitions are given below.

Definition 5.2. By a *rooted graph* we mean a finite graph G with a distinguished vertex $r(G)$. By a *rooted tree* we mean a rooted graph which is a tree. On a rooted tree we define an order \leq by $x \leq y$ if every arc containing $r(T)$ and y contains x . We require epimorphisms $f: T \rightarrow S$ between rooted trees to preserve orders, in particular $f(r(T)) = r(S)$.

Epimorphisms between rooted trees need not be confluent as the following example shows.

Example 5.3. Let S be a triod with vertices $\{A, B, C, D\}$, edges $\{\langle A, B \rangle, \langle B, C \rangle, \langle B, D \rangle\}$, and A is the root of T . Let T be another triod with vertices $\{a, b_1, b_2, c, d_1, d_2\}$, edges $\{\langle a, b_1 \rangle, \langle a, b_2 \rangle, \langle b_1, c \rangle, \langle b_1, d_1 \rangle, \langle b_2, d_2 \rangle\}$, and a is the root of S . Let $f: T \rightarrow S$ be given by $f(a) = A$, $f(b_1) = f(b_2) = B$, $f(c) = C$, and $f(d_1) = f(d_2) = D$. Then f is an order preserving epimorphism on rooted trees but is not confluent.



Definition 5.4. By an *ordered graph* G we mean finite rooted graph with a partial order \leq such that:

- the root $r(G)$ is the least element in $V(G)$ i.e. $r(G) \leq x$ for each $x \in V(G)$;
- if $\langle a, b \rangle \in E(G)$, $a \neq b$ then $a \leq b$ or $b \leq a$ and if $c \in V(G) \setminus \{a, b\}$ then $a \leq c \leq b$ is not true.

The standard amalgamation of rooted trees with order preserving epimorphisms need not be a tree as the following example shows.

Example 5.5. Let A consist of a single vertex, B and C be edges then the standard amalgamation gives a complete graph on four vertices.

Definition 5.6. By a *chain* in an ordered graph G we mean sequence a_1, a_2, \dots, a_n of vertices of G such that:

- (1) $a_1 = r(G)$;
- (2) $a_1 \leq a_2 \leq \dots \leq a_n$.

A chain is *proper* if moreover it satisfies the condition

- (3) $a_1 \neq a_2 \neq \dots \neq a_n$.

Theorem 5.7. Consider epimorphisms $f: B \rightarrow A$ and $g: C \rightarrow A$ between rooted trees (in particular preserving the orders) and suppose f is light. Then the standard amalgamation procedure gives a rooted tree D , epimorphisms $f_0: D \rightarrow B$, $g_0: D \rightarrow C$, and the epimorphism g_0 is light.

Proof. The lightness of the epimorphism g_0 is a consequence of Proposition 4.7. First we show that D is a rooted tree. We put the root of D at $((r(B), r(C)))$. Suppose on the contrary, that the graph D is not a tree. Then there are two proper chains $(b_1, c_1), (b_2, c_2) \dots (b_n, c_n)$ and $(b'_1, c'_1), (b'_2, c'_2) \dots (b'_m, c'_m)$ of vertices of D such that:

- (1) $(b_n, c_n) = (b'_m, c'_m)$;
- (2) there is an index k such that $(b_k, c_k) \neq (b'_k, c'_k)$.

We are heading toward contradiction by two Claims.

Claim 1. The chains $(b_1, c_1), (b_2, c_2), \dots (b_n, c_n)$ and $(b'_1, c'_1), (b'_2, c'_2), \dots (b'_m, c'_m)$ have the same length, i.e $n = m$, and $c_1 = c'_1, c_2 = c'_2, \dots$

$c_n = c'_n$. To see this note that the epimorphism g_0 is just the projection onto the second coordinate and it is light, therefore it is one-to-one on chains. Since there is only one chain in C joining $r(C)$ and c_n we have $c_1 = c'_1, c_2 = c'_2, \dots, c_n = c'_n$.

Claim 2. If $(b_k, c_k) \neq (b'_k, c_k)$, then b_k and b'_k are incomparable, i.e. $b_k \not\leq b'_k$ and $b'_k \not\leq b_k$. Assume on the contrary that $b_k \leq b'_k$. Then, by commutativity of the diagram (D1) we have $f(b_k) = f(b'_k) = g(c_k)$ and because f preserves orders we have also $f(b_k) = f(b_{k+1}) = \dots = f(b'_k) = g(c_k)$, which contradicts the lightness of f .

We then have a contradiction to the fact that B is a tree, since b_1, b_2, \dots, b_n and b'_1, b'_2, \dots, b'_n are two chains in B from $r(B)$ to b_n passing through two incomparable points b_k and b'_k respectively.

Finally, it follows from the definition of the standard amalgamation procedure that f_0 and g_0 preserve order. \square

Proposition 5.8. *Under the hypothesis of Proposition 4.12 if G and H are rooted trees and f is an epimorphism between rooted trees, i.e. it preserves the order, then M is a rooted tree, and the epimorphisms m and l preserve orders.*

Proof. First, observe that hereditary unicoherence of trees implies that every monotone epimorphism preserves orders. This implies that M is a rooted tree and that m is an epimorphism between rooted trees. Since f and m preserve orders, l preserves the order as well. \square

The family of rooted trees with order preserving epimorphisms were considered in [2]. The authors show there that the topological realization of the projective Fraïssé limit of the family is homeomorphic to the Lelek fan.

Observation 5.9. *If $f_H^G: G \rightarrow H$ is an epimorphism between rooted trees and C is a chain in G , then $f_H^G|_C$ is monotone.*

Theorem 5.10. *Let \mathcal{F} be a projective Fraïssé family of rooted trees and let \mathbb{T} be a projective Fraïssé limit of \mathcal{F} . For any $x, y \in \mathbb{T}$, $x \leq y$ if and only if for any $T \in \mathcal{T}$ and for any epimorphism $f_T: \mathbb{T} \rightarrow T$, $f_T(x) \leq f_T(y)$.*

Proof. Denote by \mathbb{T} the projective Fraïssé limit of \mathcal{T} . By the root $r(\mathbb{T})$ we denote the only point of \mathbb{T} such that for any tree $T \in \mathcal{T}$ and any epimorphism $f_T: \mathbb{T} \rightarrow T$ we have $f_T(r(\mathbb{T})) = r(T)$.

First assume for any epimorphism $f_T: \mathbb{T} \rightarrow T$ that $f_T(x) \leq f_T(y)$. Let I_T be the arc from $f_T(r(T))$ to $f_T(y)$. Then $f_T(x) \in I_T$ because $f_T(x) \leq f_T(y)$. The family $\mathcal{A} = \{I_T : T \in \mathcal{F}\}$ forms a projective Fraïssé family with monotone epimorphisms by Observation 5.9. By

Theorem 3.8 the projective Fraïssé limit of \mathcal{A} is an arc from $r(\mathbb{T})$ to y that contains x , so $x \leq y$ as needed.

To see the other implication let $\{T_n, \alpha_n\}$ be a fundamental sequence for \mathbb{T} . Take $x, y \in \mathbb{T}$ such that $x \leq y$ and suppose, by contradiction, that there is an m such that $x_m \not\leq y_m$. For any n let I_n be a chain in T_n from $r(T_n)$ to y_n . Then $x_m \notin I_m$. Since the epimorphisms in \mathcal{T} preserve order, we have $\mathbb{I} = \varprojlim \{I_n, \alpha_n|_{I_n}\}$ is an arc in \mathbb{T} containing y and $r(\mathbb{T})$. Since $x \leq y$, $x \in \mathbb{I}$, so $x_m \in I_m$ contrary to the assumption that $x_m \notin I_m$. For any $T \in \mathcal{T}$ and $f_T: \mathbb{T} \rightarrow T$ there is an m and a monotone epimorphism $f: T_m \rightarrow T$ such that $f_T = f \circ \alpha_m$ so $f_T(x) = f \circ \alpha_m(x) \leq f \circ \alpha_m(y) = f_T(y)$. \square

Corollary 5.11. *If \mathcal{T} is a projective Fraïssé family of rooted trees, and \mathbb{T} is a projective Fraïssé limit of \mathcal{T} , then \mathbb{T} is arcwise connected.*

Proof. By Theorem 5.10 the root $r(\mathbb{T})$ is the least element in the order \leq on \mathbb{T} , in particular there is an arc in \mathbb{T} joining $r(\mathbb{T})$ to any point in \mathbb{T} . \square

By Theorem 2.19 and Corollary 5.11 we get the following Corollary.

Corollary 5.12. *If \mathcal{T} is a projective Fraïssé family of rooted trees, and \mathbb{T} is a projective Fraïssé limit of \mathcal{T} , then \mathbb{T} is a dendroid.*

Recall the following definition for continua.

Definition 5.13. A continuum X which is a dendroid is said to be a *smooth dendroid* if there exists a point $p \in X$ such that if $x_n \in X$ is a sequence of points that converges to a point $x \in X$ then the sequence of arcs px_n converges to the arc px . This is equivalent to saying the order \leq , defined by $x \leq y$ if every arc joining p and y contains x , is closed.

We now give an analogous definition for topological rooted graph to be a smooth dendroid.

Definition 5.14. A topological rooted graph X with the root $r(X)$ is called a *smooth dendroid* if X is a dendroid according to Definition 2.11 and the order \leq , defined by $x \leq y$ if every arc joining $r(X)$ and y contains x , is closed.

Observation 5.15. *If a rooted topological graph G is a smooth dendroid and $E(G)$ is transitive, then its topological realization $|G|$ is a smooth dendroid in the topological sense.*

Corollary 5.16. *If \mathcal{T} is a projective Fraïssé family of rooted trees, and \mathbb{T} is a projective Fraïssé limit of \mathcal{T} , then \mathbb{T} is a smooth dendroid.*

Proof. First note that by Corollary 5.12 \mathbb{T} is a dendroid. It follows from Theorem 5.10 that the order \leq is closed on \mathbb{T} , so the dendroid is smooth. \square

The Lelek fan can be characterized as a smooth fan with a dense set of end points, [5]. In [2], it is shown that the topological realization of the Fraïssé limit for the family of all rooted trees with order preserving epimorphisms is the Lelek fan.

6. KELLEY TOPOLOGICAL GRAPHS

Notation 6.1. For a given metric space X with a metric d , the r -neighborhood of a closed set $A \subseteq X$ is $N(A, r) = \{x \in X : \text{there is } a \in A : d(x, a) < r\}$. The Hausdorff distance H between closed subsets of X is defined by $H(A, B) = \inf\{r > 0 : A \subseteq N(A, r) \text{ and } B \subseteq N(A, r)\}$.

Definition 6.2. A continuum X is said to be *Kelley continuum* if for every subcontinuum K of X , every $p \in K$, and every sequence $p_n \rightarrow p$ in X there are subcontinua K_n of X such that $p_n \in K_n$ and $\lim K_n = K$. By compactness, this definition is equivalent to saying that for every $\varepsilon > 0$ there is $\delta > 0$ such that for each two points $p, q \in X$ satisfying $d(p, q) < \delta$, and for each subcontinuum $K \subseteq X$ such that $p \in K$ there is a subcontinuum $L \subseteq X$ satisfying $q \in L$ and $H(K, L) < \varepsilon$.

Below we adapt the definition of Kelley continuum to the setting of topological graphs in order to prove properties of the topological realization $|\mathbb{T}_{\mathcal{C}}|$ where $\mathbb{T}_{\mathcal{C}}$ is the projective Fraïssé limit of $\mathcal{T}_{\mathcal{C}}$.

Definition 6.3. A topological graph X is called *Kelley* if X is connected and for every closed and connected set $K \subseteq V(X)$, every vertex $p \in K$, and every sequence $p_n \rightarrow p$ of vertices in X there are closed and connected sets K_n such that $p_n \in K_n$ and $\lim K_n = K$. By compactness, this definition is equivalent to saying that for every $\varepsilon > 0$ there is $\delta > 0$ such that for each two vertices $p, q \in X$ satisfying $d(p, q) < \delta$, and for each connected set $K \subseteq X$ such that $p \in K$ there is a connected set $L \subseteq X$ satisfying $q \in L$ and $H(K, L) < \varepsilon$. Note that every finite graph is Kelley.

Observation 6.4. *If a topological graph G is Kelley and $E(G)$ is transitive, then its topological realization $|G|$ is a Kelley continuum.*

Theorem 6.5. *If \mathcal{G} is a projective Fraïssé family of graphs with confluent epimorphisms, then the projective Fraïssé limit is Kelley.*

Proof. Let $\{G_n, \alpha_n\}$ be a fundamental sequence for \mathcal{G} so $\mathbb{G} = \varprojlim \{G_n, \alpha_n\}$. The proof that the inverse limit of Kelley continua with confluent bonding maps in [6, Theorem 3.1] is a Kelley continuum can be adapted to the present setting to show that \mathbb{G} is Kelley. \square

7. END VERTICES AND RAMIFICATION VERTICES IN TOPOLOGICAL GRAPHS

Definition 7.1. For a topological graph D that is a dendroid, we say that a vertex v has order at least n , in symbols $\text{ord}(v) \geq n$ if there are arcs A_1, A_2, \dots, A_n such that $A_i \cap A_j = \{v\}$ for $i, j \in \{1, 2, \dots, n\}$ satisfying $i \neq j$. We define $\text{ord}(v) = n$ if $\text{ord}(v) \geq n$ and $\text{ord}(v) \geq n+1$ is not true. If $\text{ord}(v) \geq n$ for each positive integer n , then we say that v has infinite order, in symbols $\text{ord}(v) = \infty$. Vertices of order 1 are called end vertices and vertices of order ≥ 3 are called ramification vertices.

Note that these definitions agree with Definitions 1.1 in case of finite graphs. We call an end vertex *isolated* if it belongs to an edge. Note that, according to this definition, every end vertex in a finite graph is isolated.

Let us recall a continuum theory definition.

Definition 7.2. For a continuum X a point $p \in X$ is called a *ramification point of X in the classical sense* if there are three arcs A, B, C in X such that $A \cap B = A \cap C = B \cap C = \{p\}$. Similarly, a point $p \in X$ is called an *end point of X in the classical sense* if it is an end point of every arc in X that contains p .

Proposition 7.3. *Suppose D is a topological graph that is a dendroid, the set of edges of D is transitive and e is a non-isolated end vertex of D . Let φ be the quotient map from D onto its topological realization $|D|$. Then $\varphi(e)$ is an end point of $|D|$ in the classical sense.*

Proof. Assume the conclusion is not true. Then there exists an arc $A \subseteq |D|$ such that $A \setminus \{\varphi(e)\} = H \cup K$ where H and K are non-empty separated sets. The set $\varphi^{-1}(H \cup K)$ is not connected since the image of a connected set is connected. However the sets $\varphi^{-1}(H)$ and $\varphi^{-1}(K)$ are each connected. Let a and b be vertices in $\varphi^{-1}(H)$ and $\varphi^{-1}(K)$ respectively and A_H and A_K be arcs in D from a to e and b to e . The arcs A_H and A_K then lie in $\varphi^{-1}(H) \cup \{e\}$ and $\varphi^{-1}(K) \cup \{e\}$ respectively. The intersection of A_H and A_K is $\varphi^{-1}(\varphi(e)) = \{e\}$ since e is a non-isolated end vertex of D . So $\varphi^{-1}(H) \cup \varphi^{-1}(K) \cup \{e\}$ is an

arc containing e but e is not the end vertex of this arc. Thus e is not an end vertex of D contrary to the hypothesis. \square

Proposition 7.4. *Let $f: X \rightarrow Y$ be an order preserving epimorphism between smooth dendroids. Then, for every end vertex $y \in Y$ there is an end vertex $x \in X$ such that $f(x) = y$.*

Proof. Let z be any vertex in X such that $f(z) = y$, and let x be an end vertex of X satisfying $z \leq x$. Since f is order preserving we have $f(z) = y \leq f(x)$, but y is an end, point, so $f(z) = f(x) = y$. \square

Proposition 7.5. *Suppose D is a topological graph that is a dendroid, the set of edges of D is transitive and r is a ramification vertex of D . Let φ be the quotient map from D onto its topological realization $|D|$. If $\varphi^{-1}(\varphi(r))$ contains no isolated end vertices, then $\varphi(r)$ is a ramification point of $|D|$ in the classical sense.*

Proof. Suppose r is a ramification vertex of D and let A , B , and C be arcs in D such that $A \cap B = B \cap C = C \cap A = \{r\}$. The images $\varphi(A)$, $\varphi(B)$, and $\varphi(C)$ are arcs or degenerate with pairwise intersections equaling $\varphi(r)$. Suppose $\varphi(A)$ is degenerate then A consists of a single edge $\langle r, e \rangle$ and $A = \varphi^{-1}(\varphi(r))$ contains the isolated end vertex e contrary to the hypothesis. Thus $\varphi(r)$ is a ramification point of $|D|$. \square

The following Observation follows from Theorem 5.10.

Observation 7.6. *Suppose that \mathcal{T} is a projective Fraïssé family of rooted trees and that a vertex $e \in \mathbb{T}$ has the property that for every $T \in \mathcal{T}$ and every epimorphism $f_T: \mathbb{T} \rightarrow T$ the image $f_T(e)$ is an end vertex of T . Then e is an end vertex of \mathbb{T} .*

Theorem 7.7. *Suppose that \mathcal{T} is a projective Fraïssé family of rooted trees and for every $T \in \mathcal{T}$, for every end vertex $e \in V(T) \setminus \{r(T)\}$ and every $a \in V(T)$ such that $\langle a, e \rangle \in E(T)$ there is a tree S , an epimorphism $f_T^S: S \rightarrow T$ such that for every $p \in (f_T^S)^{-1}(a)$ there are vertices $q, r \in V(S)$ satisfying $p \leq q \leq r$, $q \neq r$, and $f_T^S(q) = f_T^S(r) = e$. Then the projective Fraïssé limit of \mathcal{T} has no isolated end vertices.*

Proof. Let \mathcal{T} be a family that satisfies the assumptions of the Theorem, and let \mathbb{T} be the projective Fraïssé limit of \mathcal{T} . Suppose e is an isolated end vertex of \mathbb{T} then there exists $a \in V(\mathbb{T})$ such that $\langle a, e \rangle \in E(\mathbb{T})$. Let $T \in \mathcal{T}$ and $f_T: \mathbb{T} \rightarrow T$ be an epimorphism such that $f_T(a) \neq f_T(e)$ then $\langle f_T(a), f_T(e) \rangle \in E(T)$. From the hypothesis we know there is a tree S and an epimorphism $f_T^S: S \rightarrow T$ such that for every $p \in (f_T^S)^{-1}(f_T(a))$ there are vertices $q, r \in V(S)$ satisfying $p \leq q \leq r$, $q \neq r$,

and $f_T^S(q) = f_T^S(r) = f_T(e)$. Let $f_S: \mathbb{T} \rightarrow S$ be an epimorphism such that $f_T = f_T^S \circ f_S$. Then $f_S(a) \in (f_T^S)^{-1}(f_T(a))$ and $r = f_S(e)$, since f_S is an order preserving epimorphism, but these vertices are not members of an edge in S contrary to the fact that f_S is an epimorphism. \square

8. MONOTONE RETRACTIONS AS EPIMORPHISMS

This section is devoted to the family of trees with monotone retractions as epimorphisms. We will show the the family is a projective Fraïssé family, we will prove a characterization of its projective Fraïssé limit, but the set of edges is not transitive, so there is no topological realization of the limit.

First, let us make an important observation that allows us to use the results from previous sections.

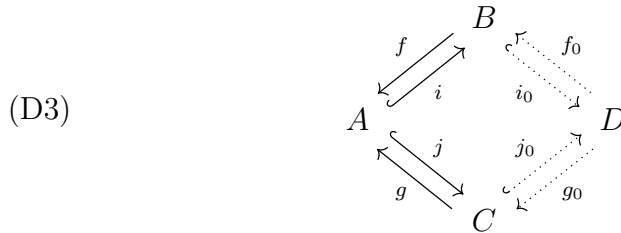
Observation 8.1. *Any monotone epimorphism between rooted trees is order preserving.*

Definition 8.2. An epimorphism $f: F \rightarrow G$ between topological graphs is called a *retraction* if there is a monomorphism $i: G \rightarrow F$ such that the composition $f \circ i$ is the identity on G . One may think of G as a subgraph of F and i as the inclusion.

Definition 8.3. The symbol \mathcal{T}_{MR} denotes the family of trees with monotone retractions as epimorphisms.

Theorem 8.4. *The family \mathcal{T}_{MR} is a projective Fraïssé family.*

Proof. Again, conditions (1) and (2) are obvious, condition (3) follows from (4) if we take the one vertex graph as A and constant epimorphisms as f and g . So we only need to show the amalgamation condition (4). Suppose trees A , B , and C , epimorphisms f and g of diagram (D1) are given. We may assume that B and C are disjoint. Let $i: A \rightarrow B$ and $j: A \rightarrow C$ be such that $f \circ i = \text{id}_B$ and $g \circ j = \text{id}_C$ (see the diagram (D3)).



Let D' be the disjoint union of $B \cup C$, and define an equivalence relation \sim on D' in the following way: if $b \in V(B)$ and $c \in V(C)$, then $b \sim c$ if and only if $b = i(p)$ and $c = j(p)$ for some $p \in V(A)$. Define \sim as the

smallest equivalence relation that satisfies the above condition. Finally, define $D = D' / \sim$, $f_0|_B = \text{id}_B$ and $f_0|_C = g|_C$ similarly, $g_0|_B = f|_B$, and $g_0|_C = \text{id}_C$. One can verify that D is a tree. To see that that f_0 and g_0 are retractions define $i_0: B \rightarrow D$ and $j_0: C \rightarrow D$ by $i_0(b) = b$ for $b \in V(B)$ and $j_0(c) = c$ for $c \in V(C)$ and note that i_0 and j_0 send edges to edges, and that $f_0 \circ i_0 = \text{id}_B$ and $g_0 \circ j_0 = \text{id}_C$. \square

Observation 8.5. *The monotone epimorphisms f_G from the Fraïssé limit $\mathbb{T}_{\mathcal{MR}}$ are retractions.*

To see how the projective Fraïssé limit of $\mathcal{T}_{\mathcal{MR}}$ looks like we need the following characterization.

Theorem 8.6. *There is unique (up to isomorphisms) topological graph P that satisfies the following conditions:*

- (1) P is a dendrite;
- (2) the set of end vertices of P is dense;
- (3) the set of ramification vertices of P is dense;
- (4) each vertex of P is either an end vertex or it has infinite order;
- (5) for each two ramification vertices $a, b \in V(P)$ there is a finite sequence $a_0 = a, a_1, \dots, a_n = b$ joining a and b and such that $\langle a_i, a_{i+1} \rangle \in E(P)$ for $i \in \{0, 1, \dots, n-1\}$.

A picture of an approximation of the graph P is presented below.

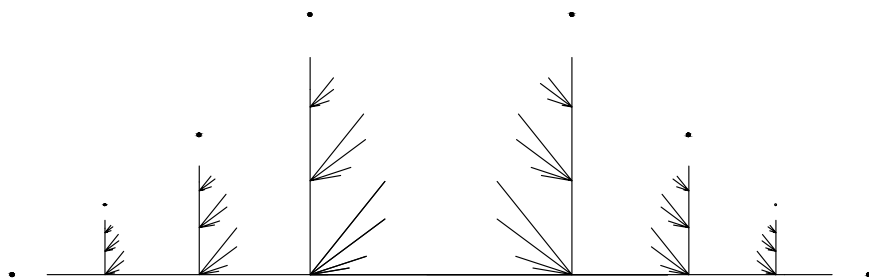


FIGURE 1. The dendrite P

Proof. Let X and Y be two topological graphs that satisfy conditions (1)–(5). We use the idea of [1, Theorem 4.5]. Choose ramification vertices $a \in V(X)$ and $b \in V(Y)$. Denote by a_1, a_2, \dots all the vertices adjacent to a and by b_1, b_2, \dots all the vertices adjacent to b . We define the isomorphism $f: X \rightarrow Y$ by steps. We start with putting $f(a) = b$ and $f(a_i) = b_i$. For a fixed index i let $a_{i,1}, a_{i,2}, \dots$ be all the vertices adjacent to a_i , except a , and let $b_{i,1}, b_{i,2}, \dots$ be all the vertices adjacent to

b_i , except b ; we put $f(a_{i,j}) = b_{i,j}$. We continue in this manner; by condition (5) each vertex of X is of the form $a_{\alpha_1, \alpha_2, \dots, \alpha_n}$ for some numbers $n, \alpha_1, \dots, \alpha_n \in \{1, 2, \dots\}$. We define indexes of the vertices of Y in the same manner, so each ramification vertex is of the form $b_{\alpha_1, \alpha_2, \dots, \alpha_n}$ for some numbers $n, \alpha_1, \dots, \alpha_n \in \{1, 2, \dots\}$ and we define the isomorphism f on ramification vertices by $f(a_{\alpha_1, \alpha_2, \dots, \alpha_n}) = b_{\alpha_1, \alpha_2, \dots, \alpha_n}$; then we extend f to end vertices by continuity. \square

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Lemma 8.7. *If $h: F \rightarrow G$ is a monotone epimorphism between dendrites and e is an end vertex of G , then $h^{-1}(e)$ contains an end vertex.*

Proof. This follows from Observation 8.1 and Proposition 7.4. \square

Lemma 8.8. *If v is a vertex in $\mathbb{T}_{\mathcal{MR}}$ which is not an end vertex, then there is a graph $G \in \mathcal{T}_{\mathcal{MR}}$, a monotone retraction $f_G: \mathbb{T}_{\mathcal{MR}} \rightarrow G$, and an inclusion $i^G: G \rightarrow \mathbb{T}_{\mathcal{MR}}$ such that $f_G \circ i^G = \text{id}_G$ and $v = i^G(f_G(v))$.*

Proof. Suppose the contrary, i.e. there is a vertex v in $\mathbb{T}_{\mathcal{MR}}$ which is not an end vertex, such that for every a graph $G \in \mathcal{T}_{\mathcal{MR}}$, every monotone retraction $f_G: \mathbb{T}_{\mathcal{MR}} \rightarrow G$, and the inclusion $i^G: G \rightarrow \mathbb{T}_{\mathcal{MR}}$, $i^G(f_G(v)) \neq v$. Then there are two arcs A and B in $\mathbb{T}_{\mathcal{MR}}$ such that $A \cap B = \{v\}$ and there is a graph G and a monotone retraction $f_G: \mathbb{T}_{\mathcal{MR}} \rightarrow G$ such that $f_G(A)$ and $f_G(B)$ are nondegenerate arcs in G . If $i^G: G \rightarrow \mathbb{T}_{\mathcal{MR}}$ is the inclusion, then $i^G(f_G(A))$ and $i^G(f_G(B))$ are subarcs of A and B respectively. This implies that $i^G(G)$ is a closed and connected subgraph of $\mathbb{T}_{\mathcal{MR}}$ intersecting both A and B . Then $A \cup B \cup i^G(G)$ is a connected, but not unicoherent subgraph of $\mathbb{T}_{\mathcal{MR}}$ contradicting the fact that $\mathbb{T}_{\mathcal{MR}}$ is a dendrite. \square

Theorem 8.9. *The projective Fraïssé limit $\mathbb{T}_{\mathcal{MR}}$ of the family $\mathcal{T}_{\mathcal{MR}}$ is isomorphic to P .*

Proof. We need to show that the Fraïssé limit $\mathbb{T}_{\mathcal{MR}}$ satisfies conditions (1) – (5) of Theorem 8.6. First, $\mathbb{T}_{\mathcal{MR}}$ is a dendrite by Corollary 3.10. We will show conditions (2) and (3) simultaneously: let $v \in \mathbb{T}_{\mathcal{MR}}$ and let $\varepsilon > 0$; we need to find an end vertex and a ramification vertex in a ε -neighborhood of v . Let F be a tree in $\mathcal{T}_{\mathcal{MR}}$ and let $f_F: \mathbb{T}_{\mathcal{MR}} \rightarrow F$ be an ε -epimorphism. Construct a tree G and a monotone retraction $f_F^G: G \rightarrow F$ such that there are vertices v_e and v_r in $V(G)$ such that $f_F^G(v_e) = f_F^G(v_r) = f_F(v)$, v_e is an end vertex while v_r is a ramification vertex of G . Let $i^G: G \rightarrow \mathbb{T}_{\mathcal{MR}}$ be, by Observation 8.5, the inclusion

monomorphism, so $i^G(v_r)$ is a ramification vertex of $\mathbb{T}_{\mathcal{MR}}$ which is in a ε -neighborhood of v , while $(f_G)^{-1}(v_e)$ contains an end vertex by Lemma 8.7, where $f_G: \mathbb{T}_{\mathcal{MR}} \rightarrow G$ is an epimorphism satisfying $f_F^G \circ f_G = f_F$.

To see condition (4) suppose v is a vertex of $\mathbb{T}_{\mathcal{MR}}$ which is not an end vertex, $\text{ord}(v) = n < \infty$, and by Observation 7.6 we may let F be a tree such that $f_F(v)$ is not an end vertex of F where, as previously, f_F is a monotone retraction of $\mathbb{T}_{\mathcal{MR}}$ onto F . Construct a tree G , a monotone retraction $f_F^G: G \rightarrow F$, and an inclusion monomorphism $i_G^F: F \rightarrow G$ such that $f_F^G \circ i_G^F = \text{id}_F$ and $\text{ord}(i_G^F(f_F(v))) > n$. Let $f_G: \mathbb{T}_{\mathcal{MR}} \rightarrow G$ be an epimorphism satisfying $f_F^G \circ f_G = f_F$ and let $i^G: G \rightarrow \mathbb{T}_{\mathcal{MR}}$ be the inclusion monomorphism. Then $\text{ord}(v) = \text{ord}(i^G(i_G^F(f_F(v)))) > n$, a contradiction.

To see condition (5), let a, b be vertices in $\mathbb{T}_{\mathcal{MR}}$, let $G \in \mathcal{T}_{\mathcal{MR}}$, let $f_G: \mathbb{T}_{\mathcal{MR}} \rightarrow G$ be a monotone retraction, and let $i^G: G \rightarrow \mathbb{T}_{\mathcal{MR}}$ be, by Lemma 8.8, the inclusion such that $f_G \circ i^G = \text{id}_G$, $a = i^G(f_G(a))$ and $b = i^G(f_G(b))$. We choose a sequence $p_0 = f_G(a), p_1, \dots, p_n = f_G(b)$ joining $f_G(a)$ and $f_G(b)$ and such that $\langle p_i, p_{i+1} \rangle \in E(G)$ for $i \in \{0, 1, \dots, n-1\}$. Then $i^G(p_0) = a, i^G(p_1), \dots, i^G(p_n) = b$ is the required sequence. \square

Remark 8.10. The set of edges of $\mathbb{T}_{\mathcal{MR}}$ is not transitive, so there is no topological realization of it.

Proof. One can show that the epimorphisms from $\mathbb{T}_{\mathcal{MR}}$ onto trees in $\mathcal{T}_{\mathcal{MR}}$ are retractions. Let G be a tree, $f_G: \mathbb{T}_{\mathcal{MR}} \rightarrow G$ be an epimorphism and $\langle a, b \rangle, \langle b, c \rangle \in E(G)$. Then there is an monomorphism $i: G \rightarrow \mathbb{T}_{\mathcal{MR}}$ such that $f_G \circ i$ is the identity map on G . So, since i is a monomorphism, $\langle i(a), i(b) \rangle, \langle i(b), i(c) \rangle \in E(\mathbb{T}_{\mathcal{MR}})$, but $\langle i(a), i(c) \rangle \notin E(\mathbb{T}_{\mathcal{MR}})$ since $\mathbb{T}_{\mathcal{MR}}$ is uncoherent. \square

9. ROOTED TREES AND EPIMORPHISM PRESERVING END VERTICES

In this section we consider order preserving epimorphisms on trees which also preserve end vertices.

Definition 9.1. We say that an epimorphism $f: S \rightarrow T$ between rooted trees *preserves end vertices* if, beside preserving order, we have that an image of an end vertex in S is an end vertex in T .

Definition 9.2. A *branch* of a rooted tree T is a maximal chain in T , the *length* of a branch is the number of vertices in the branch.

Definition 9.3. A rooted tree T is a *fan* if for every $s, t \in T$ which are incomparable in the order, if $p \notin \{s, t\}$ such that $\langle p, s \rangle$ and $\langle p, t \rangle$ are in $E(T)$ the p is the root of T .

Definition 9.4. A fan F is *uniform* if the length of all branches in F are equal.

The following observation was noted in [2, Remark 2.2].

Observation 9.5. *For any rooted tree T there is a uniform fan S and an epimorphism preserving end vertices $f: S \rightarrow T$.*

Observation 9.6. *For any uniform fan F and a number n bigger than the length of branches in F there is a uniform fan F' and an epimorphism preserving end vertices $f: F' \rightarrow F$ such that the lengths of all branches in F' are bigger than n .*

Definition 9.7. Denote by \mathcal{T}_ε the family of rooted trees and epimorphisms preserving end vertices. Similarly, \mathcal{F}_ε is the family of uniform fans and epimorphisms preserving end vertices.

Theorem 9.8. *The families \mathcal{T}_ε and \mathcal{F}_ε are projective Fraïssé families and their Fraïssé limits are isomorphic.*

Proof. For amalgamation in \mathcal{T}_ε first move to \mathcal{F}_ε by Observation 9.5, then pass to fans with the same lengths of branches by Observation 9.6, and then amalgamate the fans. By Observation 9.5 the family \mathcal{F}_ε is cofinal in \mathcal{T}_ε , so their limits are isomorphic. \square

Theorem 9.9. *The projective Fraïssé limit \mathbb{F}_ε of \mathcal{F}_ε is a smooth dendroid with only one ramification vertex, namely $r(\mathbb{F}_\varepsilon)$.*

Proof. Suppose on the contrary that there is vertex $q \in \mathbb{F}_\varepsilon \setminus r(\mathbb{F}_\varepsilon)$ that is a ramification vertex. Then there are arcs A , B , and C such that $A \cap B = A \cap C = B \cap C = \{q\}$. Choosing ε small enough and an ε -epimorphism $f_G: \mathbb{F}_\varepsilon \rightarrow G$ we have that $f_G(A)$, $f_G(B)$, and $f_G(C)$ are three connected subsets of $G \setminus r(G)$ such that $f_G(A) \not\subseteq f_G(B) \cup f_G(C)$, $f_G(B) \not\subseteq f_G(A) \cup f_G(C)$, and $f_G(C) \not\subseteq f_G(B) \cup f_G(A)$. This contradicts the fact that G is a fan. \square

Let us recall the notion of a dense in itself set.

Definition 9.10. A metric space (X, d) is called *dense in itself* if for every point $p \in X$ and for every $\varepsilon > 0$ there is a point $q \in X$, $q \neq p$ such that $d(p, q) < \varepsilon$.

Theorem 9.11. *A vertex $e \in \mathbb{T}_\varepsilon$ is an end vertex if and only if for every $T \in \mathcal{T}_\varepsilon$ and every epimorphism $f_T: \mathbb{T}_\varepsilon \rightarrow T$ the image $f_T(e)$ is an end vertex of T .*

Proof. One direction is just Observation 7.6. To see the other implication, suppose on the contrary that e is an end vertex of \mathbb{T}_ε ,

$f_W: \mathbb{T}_\mathcal{E} \rightarrow W$ is an epimorphism and $f_W(e)$ is not an end vertex of W .

Consider an arc A in W that satisfies the following conditions:

- (1) $f_W(e) \in A$;
- (2) $f_W(e) \leq x$ for every $x \in A$;

By [7, Theorem 2.4] there is a sequence $T_1 = W, T_2, \dots$ of members of $\mathcal{T}_\mathcal{E}$ and epimorphisms f_1, f_2, \dots with $f_i: T_{i+1} \rightarrow T_i$ such that the inverse limit space $\varprojlim \{T_i, f_i\}$ is isomorphic to $\mathbb{T}_\mathcal{E}$. We define, by induction, a sequence A_1, A_2, \dots of arcs of T_1, T_2, \dots , with end vertex $f_{T_i}(e)$ and e_i such that the following holds:

- (1) $A_1 = A$;
- (2) $f_i(A_{i+1}) = A_i$;
- (3) $f_{T_i}: \mathbb{T}_\mathcal{E} \rightarrow T_i$ is an epimorphism such that $f_{T_i} = f_i \circ f_{T_{i+1}}$;
- (4) $f_{T_{i+1}}(e) \leq x \leq e_{i+1}$ for every $x \in A_{i+1}$.
- (5) $f_i|_{A_{i+1}}$ is a monotone epimorphism;

Assume A_i 's and e_i 's for $1 \leq i < n$ are found so that these conditions are satisfied. Let C be the component of $f_n^{-1}(A_n)$ that contains $f_{T_{n+1}}(e)$, and define $D = \{x \in C : f_{T_{n+1}}(e) \leq x\}$. Since $f_n(f_{T_{n+1}}(e))$ is not an end vertex of T_n , the set D is nondegenerate. Let e_{n+1} be an end vertex in D and define A_{n+1} to be the arc from e_{n+1} to $f_{T_{n+1}}(e)$. Note that e_{n+1} is an end vertex of T_{n+1} ; since the epimorphisms f_n preserve end vertices we have $f_n(e_{n+1}) = e_n$. Then the image $f_n(A_{n+1}) = A_n$ and conditions (1)-(5) are satisfied for $i \leq n$.

Finally, by Theorem 5.10, $\varprojlim \{A_n, f_n|_{A_{n+1}}\}$ is a monotone arc with end vertices e and $\langle e_1, e_2, \dots \rangle$, contrary to the fact that e was an end vertex of $\mathbb{T}_\mathcal{E}$. \square

Theorem 9.12. *The set of end vertices of the projective Fraïssé limit $\mathbb{T}_\mathcal{E}$ of $\mathcal{T}_\mathcal{E}$ is dense in itself.*

Proof. Let p be an end vertex of $\mathbb{T}_\mathcal{E}$, let $\varepsilon > 0$, let $G \in \mathcal{T}_\mathcal{E}$ be a tree, and let $f_G: \mathbb{T}_\mathcal{E} \rightarrow G$ be an ε -epimorphism. Observe that, by Theorem 9.11 the image $f_G(p)$ is an end vertex of G . Define a tree H which is the one point union of two copies G_1 and G_2 of G such that $r(H) = r(G_1) = r(G_2)$. Define p_1 and p_2 as end vertices in G_1 and G_2 that correspond to the vertex $f_G(p)$. Let $f_G^H: H \rightarrow G$ be the natural identification epimorphism and let $f_H: \mathbb{T}_\mathcal{E} \rightarrow H$ be an epimorphism such that $f_G = f_G^H \circ f_H$ and $f_H(p) = p_1$. By Proposition 7.4 there is an end vertex $q \in (f_H)^{-1}(p_2)$ and we have that both p and q are end vertices in $\mathbb{T}_\mathcal{E}$ and $d(p, q) < \varepsilon$ as needed. \square

Theorem 9.13. *The set of end vertices of the projective Fraïssé limit $\mathbb{T}_\mathcal{E}$ of $\mathcal{T}_\mathcal{E}$ is closed.*

Proof. Denote the set of end vertices of a topological tree T as $EV(T)$. It follows from Theorem 9.11 that $EV(\mathbb{T}_\mathcal{E}) = \bigcap \{f^{-1}(EV(T)) : T \in \mathcal{T}_\mathcal{E} \text{ and } f: \mathbb{T}_\mathcal{E} \rightarrow T \text{ is an epimorphism}\}$ showing that the set of end vertices of $\mathbb{T}_\mathcal{E}$ is closed. \square

Theorem 9.14. *The topological realization $|\mathbb{T}_\mathcal{E}|$ of $\mathbb{T}_\mathcal{E}$ is the Cantor fan.*

Proof. Collecting results of this section we see that the set of end vertices of $\mathbb{T}_\mathcal{E}$ is closed and dense in itself, i.e. a perfect set and $\mathbb{T}_\mathcal{E}$ is a smooth fan since $\mathbb{T}_\mathcal{E}$ is isomorphic to $\mathbb{F}_\mathcal{E}$ and $\mathbb{F}_\mathcal{E}$ is a smooth fan.

By Corollary 5.16 $|\mathbb{T}_\mathcal{E}|$ is a smooth dendroid. It follows from Theorems 7.7 and 7.5 that $\varphi(r(\mathbb{T}_\mathcal{E}))$, where φ is the quotient map, is a ramification point of $|\mathbb{T}_\mathcal{E}|$ and from Observation 2.14 that $\varphi(r(\mathbb{T}_\mathcal{E}))$ is the only ramification point of $|\mathbb{T}_\mathcal{E}|$ hence $|\mathbb{T}_\mathcal{E}|$ is a smooth fan. That the set of end points of $|\mathbb{T}_\mathcal{E}|$ is perfect follows from the continuity of φ . The Cantor fan is characterized by being a smooth dendroid with a perfect set of end points so $|\mathbb{T}_\mathcal{E}|$ is a Cantor fan as claimed. \square

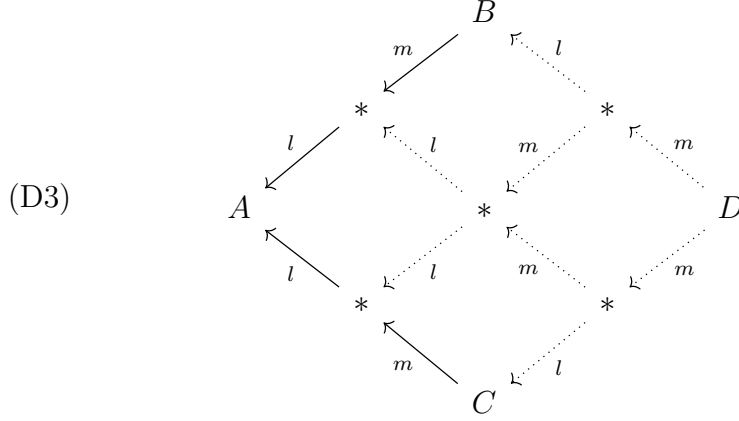
10. CONFLUENT EPIMORPHISMS AND ROOTED TREES

In this section we will use results obtained in the previous sections to investigate the family of rooted trees with confluent order preserving epimorphisms.

Notation 10.1. The family of all rooted trees and confluent order preserving epimorphisms is denoted by $\mathcal{T}_\mathcal{C}$.

Theorem 10.2. *The family of all rooted trees and confluent epimorphisms $\mathcal{T}_\mathcal{C}$ is a projective Fraïssé family.*

Proof. We need to verify condition (4) of Definition 1.2, the other conditions are straightforward.



To do so, apply the monotone-light factorization, Theorem 4.12, to functions f and g of diagram (D1). Then working the diagram (D3) above from left to right, closing the diagrams by using Theorem 5.7 if one of the epimorphism is light or Theorem 4.8 if one is monotone. Finally we can use Theorem 3.11 when both of the epimorphisms are monotone to construct the needed tree D . Note that all considered epimorphisms in the diagram (D3) are confluent by Theorems 4.14 and 4.9.

□

Theorem 10.3. *Let $\mathbb{T}_{\mathcal{C}}$ be the projective Fraïssé limit of the family $\mathcal{T}_{\mathcal{C}}$. A vertex $r \in \mathbb{T}_{\mathcal{C}}$ is either an end vertex or a ramification vertex of $\mathbb{T}_{\mathcal{C}}$.*

Proof. Suppose a vertex $r \in \mathbb{T}_{\mathcal{C}}$ is not an end vertex. Then there are vertices $a, b \in V(\mathbb{T}_{\mathcal{C}})$ such that $a \leq r \leq b$ and $a \neq r \neq b$. Let $0 < \varepsilon < \min\{d(a, r), d(r, b), d(a, b)\}$, $Z \in \mathcal{T}_{\mathcal{L}\mathcal{C}}$, and epimorphism $f_Z: \mathbb{T}_{\mathcal{L}\mathcal{C}} \rightarrow Z$ be an ε -map. Define a tree W by $V(W) = V(Z) \cup \{d\}$ and $E(W) = E(Z) \cup \{\langle f_Z(r), d \rangle\}$, and an epimorphism $f_Z^W: W \rightarrow Z$ by $f_Z^W(x) = x$ if $x \in Z$ and $f_Z^W(d) = f_Z(r)$. Let $f_W: \mathbb{T}_{\mathcal{C}} \rightarrow W$ be an epimorphism such that $f_Z = f_Z^W \circ f_W$ and let C be the component of $(f_W)^{-1}(\{f_W(r), d\})$; by confluence of f_W , see Theorem 4.4, $f(C) = \{f_W(r), d\}$, so there is an arc A in $\mathbb{T}_{\mathcal{C}}$ with r as one of its vertices, such that the three arcs: A, ra, rb witness that r is a ramification vertex of $\mathbb{T}_{\mathcal{C}}$. □

Proposition 10.4. *Let $\mathbb{T}_{\mathcal{C}}$ be the projective Fraïssé limit of the family $\mathcal{T}_{\mathcal{C}}$. Then the set of end vertices is dense in $\mathbb{T}_{\mathcal{C}}$.*

Proof. Let $p \in \mathbb{T}_{\mathcal{C}}$ and let $\varepsilon > 0$ be given. We want to find an end vertex $e \in \mathbb{T}_{\mathcal{C}}$ such that $d(p, e) < \varepsilon$. Let G be a tree and $f_G: \mathbb{T}_{\mathcal{C}} \rightarrow G$ be an ε -map. Define a tree H such that $V(H) = V(G) \cup \{q\}$ and $E(H) = E(G) \cup \{\langle f_G(p), q \rangle\}$, and let f_G^H be an epimorphism defined

by $f_G^H|_{V(G)}$ is the identity and $f_G^H(q) = f_G(p)$. Let $f_H: \mathbb{T}_{\mathcal{C}} \rightarrow H$ be an epimorphism such that $f_G^H \circ f_H = f_G$. By Proposition 7.4 there is $e \in \mathbb{T}_{\mathcal{C}}$ which is an end vertex and $f_H(e) = q$. Then $d(p, e) < \varepsilon$ since $f_G(p) = f_G(e)$ and f_G is an ε -map. □

We summarize the results of this section in the following Theorem.

Theorem 10.5. *The projective Fraïssé limit $\mathbb{T}_{\mathcal{C}}$ of the family $\mathcal{T}_{\mathcal{C}}$ has transitive set of edges and its topological realization $|\mathbb{T}_{\mathcal{C}}|$ is a Kelley dendroid (thus smooth) with a dense set of end points and such that each point is either an end point or a ramification point in the classical sense.*

Proof. Proceeding as in the proof of Theorem 3.12 we may show that the family $\mathcal{T}_{\mathcal{C}}$ satisfies the hypothesis of Theorem 2.18. Hence the set of edges is transitive. Knowing that $E(\mathcal{T}_{\mathcal{C}})$ is transitive we see from Corollary 5.16, Theorem 6.5 and Observation 6.4 that the topological realization $|\mathbb{T}_{\mathcal{C}}|$ is Kelley.

To see that the set of end points is dense in $|\mathbb{T}_{\mathcal{C}}|$ assume there exists an open set U in $|\mathbb{T}_{\mathcal{C}}|$ which contains no end points. The set $\varphi^{-1}(U)$, where φ is the quotient map, is open in $\mathbb{T}_{\mathcal{C}}$. So by Proposition 10.4 there is an end vertex $e \in \varphi^{-1}(U)$. It follows from Theorem 7.7 that $\mathbb{T}_{\mathcal{C}}$ has no isolated end vertices. Then by Proposition 7.3, $\varphi(e)$ is an end point in U contrary to our assumption. Hence the set of end points is dense in $|\mathbb{T}_{\mathcal{C}}|$.

To complete the argument we need to show that every point in $|\mathbb{T}_{\mathcal{C}}|$ is either an end point or a ramification point. Note that if p is a point in $|\mathbb{T}_{\mathcal{C}}|$ then $\varphi^{-1}(p)$ contains at most two vertices otherwise $\mathbb{T}_{\mathcal{C}}$ would not be hereditarily unicoherent. If $\varphi^{-1}(p)$ is a single vertex q then, by Proposition 10.3, q is an end vertex or a ramification vertex of $\mathbb{T}_{\mathcal{C}}$. If q is an end vertex then Proposition 7.3 and Theorem 9.9 shows that p is an end point of $|\mathbb{T}_{\mathcal{C}}|$. If q is a ramification vertex then Proposition 7.5 tells us that p is a ramification vertex of $|\mathbb{T}_{\mathcal{C}}|$. Finally, if $\varphi^{-1}(p) = \{q, r\}$ then it follows from Proposition 7.7 that both of these vertices must be ramification vertices and hence, by Proposition 7.5, that p is a ramification point of $|\mathbb{T}_{\mathcal{C}}|$. □

Problem 10.6. Give a topological characterization of the dendroid $|\mathbb{T}_{\mathcal{C}}|$.

11. CONFLUENT EPIMORPHISMS PRESERVING END VERTICES

In this section we consider the family $\mathcal{T}_{\mathcal{CE}}$ of rooted trees and confluent, order preserving epimorphisms that also preserve end vertices of the trees.

As seen in Example 5.5 above or Example 11.2 below, when the standard amalgamation procedure is used on rooted trees the resulting graph may not be a tree. Next we will describe a modification of the standard amalgamation procedure, which we call the *standard tree amalgamation*, that does yield a rooted tree.

Definition 11.1. Suppose A , B and C are rooted trees. If D' is the graph obtained via the standard amalgamation procedure then D' can be viewed as a rooted partially ordered graph where $r(D') = (r(B), r(C))$ and a partial order on D' is given by $(b_1, c_1) \leq (b_2, c_2)$ if $b_1 \leq b_2$ and $c_1 \leq c_2$.

We obtain a rooted tree D derived from D' by the following procedure. Let $V(D)$ be the set of all proper chains in D' (see Definition 5.6), $\langle c_1, c_2 \rangle \in E(D)$ if and only if when $c_1 \subseteq c_2$ then $c_2 \setminus c_1$ has at most one vertex or when $c_2 \subseteq c_1$ then $c_1 \setminus c_2$ has at most one vertex. The root of D is the chain containing just the one vertex $r(D')$. Further, a partial order on D is given by $c_1 \leq c_2$ if $c_1 \subseteq c_2$. Note that D contains no cycles. Hence D is a rooted tree.

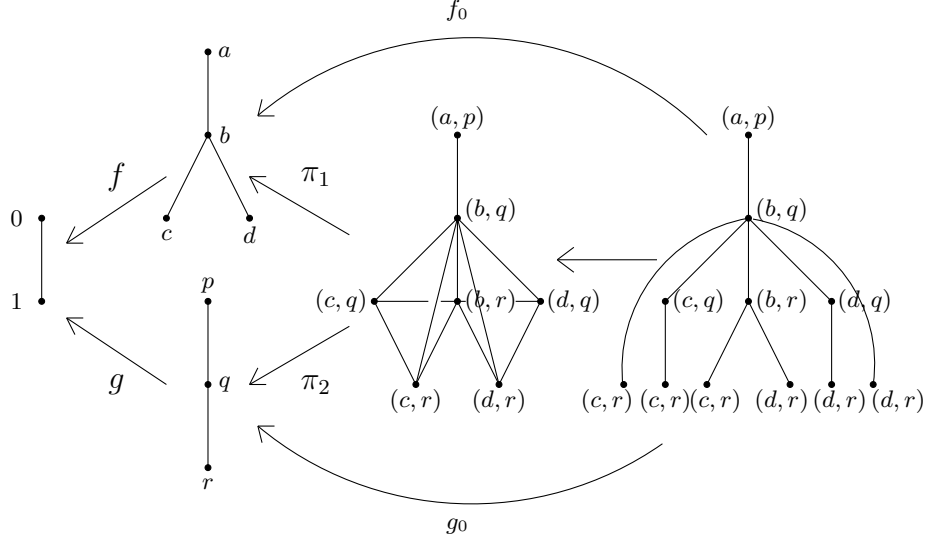
For $c \in V(D)$ define $f_0: D \rightarrow B$ and $g_0: D \rightarrow C$ by $f_0(c) = \pi_1(\max(c))$ and $g_0(c) = \pi_2(\max(c))$. Clearly then $f \circ f_0 = g \circ g_0$ so D is an amalgamation of the rooted trees B and C . One should note that if the standard amalgamation, D' , of a pair of rooted trees is a rooted tree then the standard tree amalgamation, D , of the pair is isomorphic to D' .

Note that f_0 and g_0 are order preserving epimorphisms.

The whole procedure can be seen in the following example.

Example 11.2. Let A be an edge with the two vertices 0 and 1, let B be a triod with four vertices a, b, c, d , and let C be an arc with three vertices p, q, r ; edges in all graphs are pictured below. The function $f: B \rightarrow A$ is defined by $f(a) = 0$ and $f(b) = f(c) = f(d) = 1$; similarly, $g: C \rightarrow A$ is defined by conditions $g(p) = 0$, $g(q) = g(r) = 1$. In the picture below you can find the standard amalgamation D' , that is not a tree and standard tree amalgamation D defined above. In the standard tree amalgamation the vertices are labeled by the end vertices of the chains, so for example, there are three vertices labeled (c, r) because there are three chains in D' from the root (a, p) to the vertex (c, r) .

Note that in this example the epimorphisms f and g are monotone, but the epimorphisms f_0 and g_0 are not, therefore this amalgamation procedure could not be used in Theorem 3.11



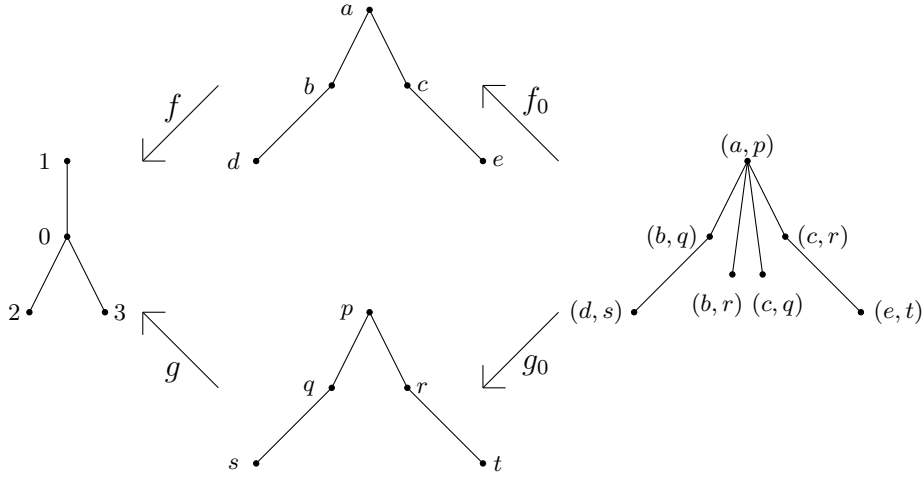
Proposition 11.3. *If in the standard tree amalgamation f and g are confluent and preserve end vertices then f_0 and g_0 preserve end vertices.*

Proof. Let e be an end vertex in D then $(b_1, c_1) \leq (b_2, c_2) \leq \dots \leq (b_n, c_n) = e$ is a maximal proper chain in D' . Suppose $f_0(e) = b_n$ is not an end vertex of B . Then there is a vertex $b_{n+1} \in V(B) \setminus \{b_n\}$ such that $b_n \leq b_{n+1}$ and $\langle b_n, b_{n+1} \rangle \in E(B)$. If $f(b_{n+1}) = f(b_n)$ then $(b_1, c_1) \leq (b_2, c_2) \leq \dots \leq (b_n, c_n) \leq (b_{n+1}, c_n)$ is a chain in D' containing e contradicting the maximality of e .

So f maps the edge $\langle b_n, b_{n+1} \rangle$ to the edge $P = \langle f(b_n), f(b_{n+1}) \rangle$ in A . Because g is confluent we know, by (3) of Theorem 4.3, that the component of $g^{-1}(P)$ that contains c_n contains vertices $c_{n+1}, \dots, c_m, c_{m+1}$ such that $c_n \leq c_{n+1} \leq \dots \leq c_m \leq c_{m+1}$ and $g(c_n) = g(c_{n+1}) = \dots = g(c_m)$ and $g(c_{m+1}) = f(b_{n+1})$. Thus we have the chain $(b_1, c_1) \leq (b_2, c_2) \leq \dots \leq (b_n, c_n) \leq (b_n, c_{n+1}) \leq (b_n, c_m) \leq (b_{n+1}, c_{m+1})$ in D' that contains e again contradicting the maximality of e . So $f_0(e)$ is an end vertex of B . A similar argument shows that $g_0(e)$ is an end vertex of C . \square

Example 11.4. In the standard tree amalgamation if f and g preserve end vertices it may not be the case that f_0 and g_0 preserve end vertices. To see this let A be a triod with end vertices 1, 2, 3, ramification vertex 0, and let 1 be the root of A . Let B and C both be arcs consisting of 5 vertices, d, b, a, c, e and s, q, p, r, t with the end vertices of B being d, e , the end vertices of C being s, t , and the roots of B and C being

$r(B) = a$ and $r(C) = p$. Finally, define $f(a) = g(p) = 1$, $f(b) = f(c) = g(q) = g(r) = 0$, $f(d) = g(s) = 2$, and $f(e) = g(t) = 3$. So f and g are end vertex preserving. The standard amalgamation procedure results in D' being a 4-od with the root (a, p) which is also the ramification vertex of the 4-od. Since, in this case, the rooted tree D obtained by the standard tree amalgamation is isomorphic to D' we just consider D' . Two of the branches of D' are the edges $\langle (a, p), (b, r) \rangle$ and $\langle (a, p), (c, q) \rangle$. So (b, r) and (c, q) are end vertices in D' but f_0 and g_0 applied to these vertices are not end vertices in B and C . Note that here f and g are not confluent.



Proposition 11.5. *Using the standard tree amalgamation procedure and the notation of diagram (D1), if the epimorphism f is confluent, then the epimorphism g_0 is confluent.*

Proof. We will use the characterization in Theorem 4.3 (2) to show that g_0 is confluent. Let $\langle c, c' \rangle \in E(C)$ and let $d = \{(b_1, c_1), \dots, (b_n, c_n)\}$ be a vertex in D that such that $g_0(d) = c$, i.e. $c_n = c$. We need to find vertices $d', d'' \in V(D)$ such that $g_0(d') = c$, $g_0(d'') = c'$, $\langle d', d'' \rangle \in E(D)$, and d, d' are in the same component of $(g_0)^{-1}(c)$. Because f is confluent, by Theorem 4.3 (2) there is an edge $\langle b', b'' \rangle \in E(B)$ such that

- (1) $f(b') = f(b_n) = g(c)$;
- (2) $f(b'') = g(c')$;
- (3) b', b_n are in the same component of $f^{-1}(f(b_n))$.

Let $b_{n+1}, \dots, b_m \in V(B)$ be such that $b_m = b'$ and $\langle b_i, b_{i+1} \rangle \in E(B)$ for $i \in \{n, \dots, m-1\}$. We may define

$$d' = (b_1, c_1), \dots, (b_n, c_n), (b_{n+1}, c_n), \dots, (b_m, c_n),$$

$$d'' = (b_1, c_1), \dots, (b_n, c_n), (b_{n+1}, c_n), \dots, (b_m, c_n), (b'', c')$$

and verify that such defined chains, points of D , satisfy all the required conditions. \square

The following theorem follows from Propositions 11.3 and 11.5.

Theorem 11.6. *The family $\mathcal{T}_{\mathcal{CE}}$ is a projective Fraïssé family.*

The following Proposition is analogous to Proposition 10.3, but this time we need to have light epimorphisms.

Proposition 11.7. *Let $\mathbb{T}_{\mathcal{CE}}$ be the projective Fraïssé limit of the family $\mathcal{T}_{\mathcal{CE}}$. A vertex $r \in \mathbb{T}_{\mathcal{CE}}$ is either an end vertex or a ramification vertex of $\mathbb{T}_{\mathcal{CE}}$.*

Proof. Suppose a vertex $r \in \mathbb{T}_{\mathcal{CE}}$ is not an end vertex. Then the set $P = \{x \in \mathbb{T}_{\mathcal{CE}} : r \leq x\}$ is a connected nondegenerate subtree of $\mathbb{T}_{\mathcal{CE}}$. Let $0 < \varepsilon < \text{diam}(P)$, and let an epimorphism $f_Z: \mathbb{T}_{\mathcal{CE}} \rightarrow Z$ be an ε -map thus $f_Z(P)$ is nondegenerate.

Define a tree W by $V(W) = V(Z) \times \{1, 2, 3\} / \sim$, where $(x, i) \sim (y, j)$ if and only if $x = y$ and $x \notin P$ or $x = f_Z(r)$ and $\langle (x, i), (y, i) \rangle \in E(W)$ if and only if $\langle x, y \rangle \in E(Z)$, with $f_Z^W: W \rightarrow Z$ being just the projection. In other words, W is the graph Z with three copies of $f_Z(P)$ attached at $f_Z(r)$.

By [7, Theorem 2.4] there is a sequence $T_1 = W, T_2, \dots$ of members of $\mathcal{T}_{\mathcal{CE}}$ and epimorphisms f_1, f_2, \dots with $f_i: T_{i+1} \rightarrow T_i$ such that the inverse limit space $\varprojlim \{T_i, f_i\}$ is isomorphic to $\mathbb{T}_{\mathcal{CE}}$. We define, by induction, a sequence $A_1^j, A_2^j, \dots, j \in \{1, 2, 3\}$ of subtrees of T_1, T_2, \dots , respectively such that the following holds for $j \in \{1, 2, 3\}$:

- (1) $A_1^j = P \times \{j\}$;
- (2) $f_i(A_{i+1}^j) = A_i^j$;
- (3) $f_{T_i}: \mathbb{T}_{\mathcal{CE}} \rightarrow T_i$ is an epimorphism such that $f_{T_i} = f_i \circ f_{T_{i+1}}$;
- (4) $A_i^j \cap A_i^k = \{f_{T_i}(r)\}$ for $k \neq j$;

Assume that $\{A_1^j, \dots, A_n^j\}$ satisfy these conditions. Then, since the epimorphisms in $\mathcal{T}_{\mathcal{CE}}$ are confluent, define A_{i+1}^j as components of $(f_i)^{-1}(A_i^j)$ containing the point $f_{T_{i+1}}(r)$. One can verify that conditions (1)-(4) are satisfied. Then the trees $A^j = \varprojlim \{A_i^j, f_i|_{A_{i+1}^j}\}$ satisfy $A^j \cap A^k = \{r\}$, for $j \neq k$, so r is a ramification vertex of $\mathbb{T}_{\mathcal{CE}}$. \square

Theorem 11.8. *The projective Fraïssé limit of $\mathcal{T}_{\mathcal{CE}}$ has transitive set of edges and the topological realization $|\mathbb{T}_{\mathcal{E}}|$ of the limit $\mathbb{T}_{\mathcal{CE}}$ is a Kelley dendroid (thus smooth) with a closed set of end points and such that each point is either an end point or a ramification point in the classical sense.*

Proof. The results used in Theorem 10.5 to show that $\mathbb{T}_{\mathcal{C}}$ has a transitive set of edges and its topological realization is a Kelley dendroid only required that the family $\mathbb{T}_{\mathcal{C}}$ consisted of rooted trees with confluent order-preserving epimorphisms. Since this is also true for $\mathbb{T}_{\mathcal{CE}}$ we see that $\mathbb{T}_{\mathcal{CE}}$ has a transitive set of edges and $|\mathbb{T}_{\mathcal{CE}}|$ is a Kelley dendroid.

Theorem 9.13 shows that the set of end vertices of $\mathbb{T}_{\mathcal{CE}}$ is closed. Since $\mathbb{T}_{\mathcal{CE}}$ is compact and φ is continuous it follows that $\varphi(EP(\mathbb{T}_{\mathcal{CE}}))$ is closed. Using an argument similar to that in Theorem 10.5 we can see that only end vertices map to end points under φ so $EP(|\mathbb{T}_{\mathcal{CE}}|) = \varphi(EP(\mathbb{T}_{\mathcal{CE}}))$.

Finally, that $|\mathbb{T}_{\mathcal{CE}}|$ contains only end points or ramification points follows from Theorem 11.7 in the same manner as in Theorem 10.5. \square

The class of dendroids known as Mohler-Nikiel dendroids, first constructed in [10], have all of the properties $|\mathbb{T}_{\mathcal{CE}}|$ was just shown to possess. The authors do not know if $|\mathbb{T}_{\mathcal{CE}}|$ is homeomorphic to a Mohler-Nikiel dendroid. This leads to the following problem.

Problem 11.9. Give a topological characterization of the dendroid $|\mathbb{T}_{\mathcal{CE}}|$.

We are currently working on two more articles in the subject: one on homogeneity properties of Fraïssé limits and a second on the Fraïssé family of all graphs with confluent epimorphisms.

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