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Thermal buckling and postbuckling of columns accounting for temperature effect on material properties

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ABSTRACT

Thermal buckling and postbuckling of columns are analyzed accounting for the effect of temperature on "material constants," i.e., the modulus of elasticity, the coefficient of thermal expansion, and the yield stress. The columns are simply supported, and fully or elastically constrained in the axial direction. Solutions are obtained by two methods. One of these methods is using "real-time" values of material constants corresponding to buckling and postbuckling at prescribed temperatures and neglecting the history of loading. The other method takes the effect of the history of loading into account using polynomial property-temperature relationships throughout the evolution of temperature. It is demonstrated that buckling temperature calculated by these methods may vary, although the trends found by them are in a qualitative agreement. Contrary to postbuckling behavior of mechanically compressed columns, in the case of thermal loading the column exhibits a noticeable resilience, i.e., temperature can significantly increase over the buckling value prior to the collapse. It is observed that since material constants are continuous functions of temperature, they can always be presented in the form of polynomials. Accordingly, the solution can be extrapolated to any material employing experimental polynomial property-temperature relationships as is illustrated in the paper.

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Axial constraint; effect of temperature on properties; history of loading; thermal buckling; thermal postbuckling

1. Introduction

Problems of thermal buckling and postbuckling of columns represent significant interest in engineering since such problems are present in numerous applications. Examples include railroad tracks [1, 2], heated pipelines [3, 4], optical fibers [5, 6], and carbon nanotubes [7, 8]. A thorough analysis of thermal buckling of columns was presented in the book by Boley and Weiner (Boley, B. A., and J. H. Weiner. 1960. Theory of Thermal Stresses, New York, Wiley).

Among the main subjects that have to be addressed to clearly comprehend and formulate the problem of thermal buckling and postbuckling of columns are:

- Effect of lateral and rotational boundary conditions. For example, it is evident that buckling of clamped columns occurs at different temperatures than that of their simply supported counterparts. An example of such study can be found in [9] where it was demonstrated that while high-strength simply supported columns experience flexural buckling, they may exhibit a flexural-torsional mode of buckling in the presence of rotational constraints. The effect of rotational constraint on thermal buckling of columns subject to fire was also studied in [10] and

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[11] where the columns were assumed completely constrained against axial displacements. A detailed study of postbuckling thermal response of columns with various boundary conditions was published by Kassimali [12].

- Effect of localized thermal loading, such as local fire. For example, this effect was investigated in [13] where it was demonstrated that such loading may cause flexural-torsional buckling.
- Effect of an elastic foundation on thermal buckling and postbuckling response may represent a significant interest in several applications (e.g., [14, 15]).
- Effect of boundary constraints on axial displacements and thermal buckling of columns. In particular, experimental study [16] emphasized that the axial constraint can be influenced by adjacent structures. It was also demonstrated that a stiffer boundary causes lower buckling temperatures of the column confirming the conclusions from other research studies (e.g., [17]).
- Accounting for the effect of temperature on material constants may be particularly important since such effect is often disregarded. Both the stiffness as well as the yield stress of engineering materials are reduced in the presence of temperature (e.g., [18–22], while the Poisson ratio may be relatively insensitive to heat [23]. In addition, thermal conductivity and the coefficient of thermal expansion are also affected by temperature. Neglecting these effects may lead to a serious mistake, except for the cases where thermal buckling and initial postbuckling response occur at temperatures much lower than the melting temperature of metallic columns or curing temperature of their composite counterparts. Effects of temperature on material properties, buckling and postbuckling of columns were studied in [24–27].
- A combination of thermal and mechanical loading (e.g., [28]).

Besides the effects outlined above, such aspects of the problem as residual stresses and initial imperfections should be mentioned. However, their influence on the response of steel columns subjected to fire were shown relatively small [29]. Creep was demonstrated to reduce thermal buckling load of metallic columns, necessitating an analysis accounting for this phenomenon [22, 30].

An early study of thermal postbuckling of a column manufactured from a physically nonlinear material was published in 1996 [31]. However, this solution did not account for geometric nonlinearity. Exact elastica solutions of the problem of thermal postbuckling of columns were presented by Coffin and Bloom [32], and Gupta et al. [11]. Vaz and Solano solved the problem of thermal buckling and initial postbuckling of a column with a nonlinear strain-temperature relationship [33, 34]; while in the first paper the simply supported column was constrained against axial displacements, in the latter paper, one boundary of the column was constrained by an elastic spring, and the other boundary was completely constrained. An interesting observation available from exact solutions of thermal postbuckling problems of simply supported columns is that the shape of the column upon buckling is virtually indistinguishable from a sinusoidal curve (e.g., [35]). Predictably, thermal buckling occurred at a higher temperature in the case of a compliant spring constraining axial displacements, as compared to more rigid springs or a complete constraint (for an experimental confirmation of this observation, see [16]).

It is important to note that large postbuckling deformations of metallic columns subject to elevated temperatures should be considered accounting for plastic effects in the material. For example, a finite element analysis accounting for the effects of temperature on the properties of typical steel columns subjected to fire demonstrated that elastic solutions can be applicable only at the deflection-to-span ratio of an order of 0.02 or less [36]. Accordingly, a valid elastic postbuckling analysis should always refer to the initial phase of the process, while a subsequent progressive postbuckling should account for the physical nonlinearity.

In the present paper, the emphasis is on accounting for the effect of temperature on material properties in thermal buckling and initial postbuckling problems of simply supported columns.

Table 1. Coefficients in property-temperature relationships (1–3) [25].

Coefficients	a_3	a_2	a_1	a_0	b_3	b_2
16Mn steel	$-3 * 10^{-9}$	$2 * 10^{-6}$	-0.0013	1.0413	$-3 * 10^{-9}$	$1 * 10^{-6}$
20MnTiB steel	$4 * 10^{-9}$	$-6 * 10^{-6}$	0.0011	0.9603	$6 * 10^{-9}$	$-8 * 10^{-6}$
Coefficients	b_1	b_0	c_1	c_0		
16Mn steel	-0.0003	1.0224	$2.9 * 10^{-8}$	$7.99 * 10^{-6}$		
20MnTiB steel	0.0016	0.9403	$2.9 * 10^{-8}$	$7.99 * 10^{-6}$		

Based on the previous solutions, flexural buckling dominates in the case of such boundary conditions (e.g., [9]). Two approaches are considered, one is using material constants corresponding to the current buckling or postbuckling temperature, neglecting the history of thermal loading, while the second accounts for the history of thermal loading. While the solutions for both thermal buckling and initial postbuckling behavior are demonstrated using the former approach, numerical analysis accounting for the history of thermal loading is limited to specifying the buckling temperature. The axial constraint can be either infinite or linear elastic. For example, an infinite constraint can be anticipated if the column models one of several identical spans of a continuous structure subjected to a uniform temperature. In such case, axial displacements of adjacent spans cancel each other at the common support due to symmetry. A finite elastic constraint may be observed in practically any other situation; neglecting such support and using solutions for fully axially constrained columns is incorrect since a fully constrained column has a lower buckling temperature than the actual structure. While the analysis is presented for two steels where polynomial property-temperature relationships are available, it can be extrapolated to an arbitrary material as is discussed in the paper. The solutions presented below are static, i.e., the changes of temperature are assumed slow, so that dynamic effects can be discounted.

2. Analysis

In this paper, we demonstrate solutions of thermal buckling and initial postbuckling problems for simply supported columns with a complete and elastic axial constraints at the boundaries. Two approaches to the analysis are entertained, one is using the real-time values of material constants and the other accounting for the history of thermal loading.

The solution is developed by assumption that material constants, such as the yield stress, the modulus of elasticity, and the coefficient of thermal expansion, can be represented by polynomial functions of temperature. Such assumption is justified by the Weierstrass Theorem that states that any continuous function within a prescribed interval can be approximated by a polynomial (Weierstrass, K. 1885 (II). Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Vanderwielen. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin).

An example of polynomial property-temperature relationships can be found in [25] for two steels broadly used in China: 16Mn steel that is employed in steel frameworks, including columns, and a less relevant to our subject 20MnTiB steel that is often used in high-strength bolts. For both steels, the yield stress was presented as

$$\sigma_y(T) = \sigma_y\left(a_3T^3 + a_2T^2 + a_1T + a_0\right) \tag{1}$$

where σ_y is the yield stress at the room (stress-free) temperature, T is the current temperature in degrees Celsius, and the coefficients a_i are specific for the material and shown for two steels considered in the paper in Table 1.

The modulus of elasticity was approximated by

$$E(T) = E\left(b_3T^3 + b_2T^2 + b_1T + b_0\right) \tag{2}$$

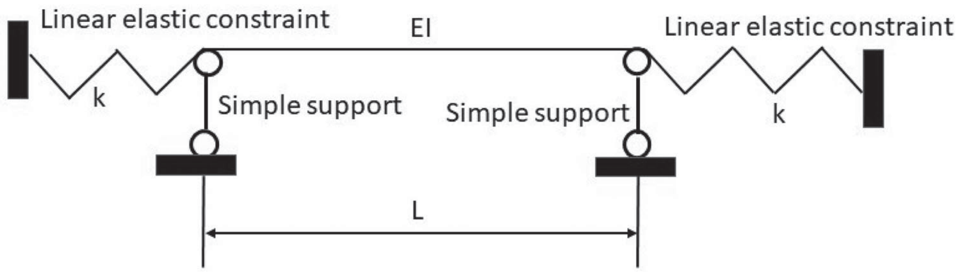

Figure 1. Problem formulation and a realistic structural arrangement where supporting beams provide simple support as well as an axial constraint to the column. The stiffness of linear elastic constraints at the boundaries of column is denoted by k. The length of the column is L.

where E is the modulus of elasticity at the room temperature, and b_j are specific material coefficients (see Table 1).

Finally, the coefficient of thermal expansion is approximated by a linear function of temperature:

$$\alpha(T) = c_1 T + c_0 \tag{3}$$

where c_n are coefficients (see Table 1).

Note that Eqs. (1)–(3) were derived from experimental data using the least-square method. While the modulus of elasticity and the coefficient of thermal expansion in relationships (2) and (3) can be used in the study accounting for the history of thermal loading, the yield stress given by Eq. (1) refers to a specific temperature value, i.e., it is not affected by the history of thermal loading. The cases of different property-temperature polynomial relationships can also be analyzed following the approach shown below.

2.1. Thermal buckling and initial postbuckling of columns with elastic axial constraints accounting for the real-time effect of temperature on material properties and neglecting the history of thermal loading

Neglecting the history of loading, i.e., the evolution of displacements, strains, and stresses with varying temperature, implies that we can modify available solutions using the values of material constants dependent on the current temperature.

Consider a simply supported column with linear-elastic axial constraints subjected to a uniform temperature (Figure 1). Irrespectively of the history of thermal loading, the geometrically nonlinear Lagrangian centroidal axial strain is given by

$$\varepsilon(x,T) = u(x,T),_x + \frac{1}{2} w(x,T),_x^2 \tag{4}$$

where $u(x,T)$ and $w(x,T)$ are the axial and transverse displacements, respectively, and x is the axial coordinate.

The axial stress in the column is obtained by

$$\sigma(x,T) = E(T)\left[\varepsilon(x,T) - \alpha(T)T\right] \tag{5}$$

The equation of axial force equilibrium is

$$N(T),_x = E(T)A\epsilon(x,T),_x - N(T),_x^T = 0 \quad \rightarrow \quad \epsilon(x,T),_x = 0 \tag{6}$$

where A is the cross-sectional area of the column.

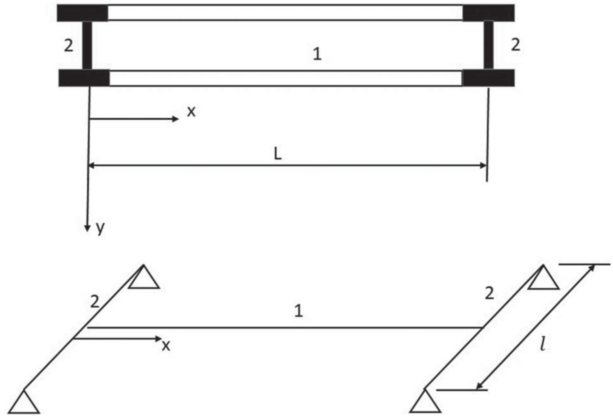

Figure 2. Model of a column (1) elastically supported against axial displacements by adjacent structures (beams 2). The length of beams 2 is l.

$$N(T)^T = E(T)\alpha(T)AT \tag{7}$$

In Eq. (6), $N(T)$ is the axial force, and the derivative of the thermal force $N(T)^T$is equal to zero since temperature does not vary along the x-coordinate.

The equation of equilibrium in the transverse direction is

$$E(T)Iw(x,T),_{xxxx} - N(T)w(x,T),_{xx} = 0 \tag{8}$$

where I is the moment of inertia of the column with respect to the centroidal axis perpendicular to the plane of deformation.

The solution of the equations of equilibrium must satisfy the boundary conditions. If the ends of the column are simply supported and elastically constrained against displacements in the axial direction, these conditions are

$$\begin{aligned} &w(x=0,\ x=L,T) = w,_{xx}(x=0,\ x=L,T) = 0, \\ &E(T)A\varepsilon(x=0,T) - N^T(T) = k(T)u(x=0,T) \\ &EA\varepsilon(x=L,T) - N^T(T) = -k(T)u(x=L,T) \end{aligned} \tag{9}$$

where L is the length of the column, and $k(T)$ is the coefficient of axial elastic constraint. The axial constraint force in Eqs. (9) is assumed proportional to the axial displacement.

Conditions (9) require some explanation. Elastic constraint is provided by adjacent to the column supporting structures (beams 2 in Figure 2). These structures, presumably manufactured from a temperature-sensitive material, will provide a constraint dependent on temperature in the structures. If the structures are insulated and thermal conduction from the column is discounted, the elastic constraint is constant and temperature-independent. In other, more realistic situations boundary structures may serve as heat sinks, but then temperature cannot be assumed constant along the entire length of the column, thus such case is not considered here. In the solution shown below, the column and supporting structures are subject to the same temperature. As a result, the solution is significantly simplified.

It is worth analyzing the validity of boundary conditions (9) in realistic structures. The condition of zero deflections at the joint between column 1 and beam 2 implies that the flexural stiffness of the beam is so large that these deflections can be neglected. As was demonstrated in a numerical example in the book (Birman, V. Plate Structures. 2011. Dordrecht. Springer), such assumption is acceptable for all but very flexible beams. The assumption that the bending moment applied to the column is equal to zero implies that torsional resistance of beams 2 is very small as is the case in beams with open profiles, such as I-beams shown in Figure 2, channel sections, and angle profiles. The axial constraint is attributed to the flexural stiffness of beam 2 in the plane perpendicular to axis y. If the column is supported by beam 2 at its midspan, such constraint generated by beam 2 can be obtained from

$$k(T) = \frac{48E(T)I_y}{l^3} \tag{10}$$

where $E(t)$ is the modulus of elasticity of the material of beam 2, I_y is the moment of inertia of beam 2 with respect to the centroidal axis y, and l is the length of the beam.

If column 1 and beams 2 are manufactured from the same material, the law governing the relationship $k(T)$ may be assumed, in the first approximation, identical with the stiffness-temperature law for the modulus of elasticity of the column, i.e.,

$$k(T) = k\left[b_3k(T)^3 + b_2k(T)^2 + b_1k(T) + b_0\right] \tag{11}$$

where k is the stiffness at the room temperature.

The solution is obtained representing the buckling mode shape by

$$\begin{aligned} u(x,T) &= U(T)\sin\frac{2\pi x}{L} + V(T)\left(1 - 2\frac{x}{L}\right) \\ w(x,T) &= W(T)\sin\frac{\pi x}{L} \end{aligned} \tag{12}$$

where $U(T)$, $V(T)$ and $W(T)$ are the amplitudes of the corresponding displacement components.

Note that if the boundaries of the column are completely constrained against axial displacements, we should set $V(T) = 0$.

Upon the substitution of Eqs. (12) into Eqs. (6) and (4), the following relationship between the amplitudes is obtained:

$$U(T) = -\frac{1}{8}\frac{\pi}{L}W(T)^2 \tag{13}$$

The boundary conditions (9) are now employed, together with Eqs. (13) and (4) yielding

$$V(T) = \frac{1}{1+2\gamma(T)}\left[\frac{\pi^2}{4L}\gamma(T)W(T)^2 - \frac{N^T(T)}{k(T)}\right] \tag{14}$$

where

$$\gamma(T) = \frac{E(T)A}{Lk(T)} \tag{15}$$

Note that if $E(T)$ and $k(T)$ change according to Eqs. (2) and (11), $\gamma(T)$ is independent of temperature, i.e., $\gamma(T) = \gamma(20^0C) = \gamma$ where 20^0C is the room temperature. This case is considered in the present solution and in the examples. If temperatures of the column and supports differ, a complicated heat transfer analysis problem should be posed and the assumption that temperature is uniform along the column becomes questionable.

The solution of the equilibrium equation (8) obtained using the Galerkin procedure yields

$$\frac{N^T(T)}{N^T_{cr}(T)} = \frac{1}{\delta(T)}\left[1 + \frac{1}{4}\lambda^2\delta(T)\overline{W(T)}^2\right] \tag{16}$$

where

$$\lambda^2 = \frac{AL^2}{I}, \quad \delta(T) = \delta = \frac{1}{1+2\gamma}, \quad \overline{W(T)} = \frac{W(T)}{L} \tag{17}$$

In Eq. (16), λ is the slenderness ratio, and $\overline{W(T)}$ is a nondimensional postbuckling deflection. The denominator in the left side of Eq. (16) represents the critical buckling force of the column that is completely constrained in the axial direction:

$$N_{cr}^T(T) = \left(\frac{\pi}{L}\right)^2 E(T)I \tag{18}$$

It is convenient to transform Eq. (16) into the relationship between the nondimensional postbuckling deflections and the postbuckling temperature:

$$T = \frac{\pi^2}{\alpha(T)\lambda^2\delta}\left[1 + \frac{1}{4}\lambda^2\delta\overline{W(T)}^2\right] \tag{19}$$

In the case of a complete axial constraint, $\gamma = 0$ and $\delta = 1$.

It is evident from Eqs. (16) and (19) that the initial postbuckling response of the column is stable, contrary to postbuckling of columns subjected to an axial compressive force. The difference may be attributed to different axial boundary conditions. Under mechanically applied compression, the boundary of the column should be free of the axial constraint, so that the applied force can be transmitted to the column. This is different from the case of thermal buckling where the boundaries of the column have to be fully or partially constrained against axial displacements to avoid a free expansion of the heated column.

The relationship between the slenderness ratio and the buckling temperature T_{cr} is available from Eq. (19):

$$\lambda^2 = \frac{\pi^2}{\alpha(T_{cr})T_{cr}\delta} \tag{20}$$

The range of validity of the present solution is limited by the assumption that the material remains linear-elastic. Thus, the stress obtained from Eq. (5) should remain smaller than the yield stress calculated by Eq. (1). This requirement is particularly easy to illustrate if the column is completely constrained against axial displacements. In such case, relating the compressive stress in the column prior to its elastic buckling to the yield stress of the material yields

$$\begin{aligned}\sigma^T(T) = E(T)\alpha(T)T = {} & E\left(b_3T^3 + b_2T^2 + b_1T + b_0\right)(c_1T + c_0)T < \sigma_y(T) \\ &= \sigma_y\left(a_3T^3 + a_2T^2 + a_1T + a_0\right)\end{aligned} \tag{21}$$

If inequality (21) is not satisfied, the stresses in the column exceed the yield limit.

For the column to remain elastic prior to buckling, inequality (21) should be satisfied for the entire range of temperature up to the buckling value. Using the buckling temperature available from Eq. (20) where $\delta = 1$ in inequality (21) we can check whether a completely constrained in the axial direction column buckles elastically.

It is also possible to predict whether the buckling temperature of a column with an axial elastic constraint remains smaller than temperature that causes yielding in such column. As follows from the linear version of Eq. (16), thermal stresses remain elastic if

$$\sigma(T_{cr}) = \frac{\pi^2}{\delta\lambda^2}E(T_{cr}) < \sigma_y(T_{cr}) = \sigma_y\left(a_3{T_{cr}}^3 + a_2{T_{cr}}^2 + a_1T_{cr} + a_0\right) \tag{22}$$

The buckling temperature can be determined from Eq. (20) as a function of the slenderness ratio. Then using this temperature in Eq. (22) one can evaluate whether the column with an axial linear-elastic constraint buckles in the elastic range.

2.2. Thermal buckling of columns accounting for the history of loading

The previous solution does not account for the history of thermal loading, i.e., the values of engineering constants are adopted at fixed temperatures neglecting the effect of a gradual evolution of deformations, strains, and stresses as temperature raises. It is useful to identify the variables affected by a gradual rise of temperature as compared to their counterparts that should be taken at a current temperature value. For example, the yield strength of the material does not evolve as an integral function of temperature, rather being a constant at each specified value, e.g., see Eq. (1). On the other hand, displacements, strains, stresses, and forces gradually increase with a rising temperature, i.e., the history of such increase could be accounted for. Likewise, such material constants as the modulus of elasticity and the coefficient of thermal expansion gradually change with temperature (e.g., Eqs. (2) and (3)).

2.2.1. Case 1: Thermal buckling of a simply supported column with a complete axial constraint on axial displacements

The value of the buckling force of a simply supported column axially constrained at the ends is not affected by the history of loading being given by Eq. (18). However, the thermally-induced compressive force is an integral function of temperature, so that instead of Eq. (7) we use

$$N(T)^T = A\int_{T_0}^{T} E(T)\alpha(T)dT \tag{23}$$

where T_0 is a reference temperature corresponding to the stress-free state of the column (e.g., 20^0C).

Noting that $N(T_0)^T = 0$, Eq. (23) results in

$$N(T)^T = A\int E(T)\alpha(T)dT \tag{24}$$

In the case where the modulus of elasticity and the coefficient of thermal expansion are given by Eqs. (2) and (3), the integration in Eq. (24) yields the following relationship between the temperature and the slenderness ratio of the column buckling at this temperature:

$$\lambda^2 = \frac{\pi^2 E(T_{cr})}{EF(T_{cr})} \tag{25}$$

where

$$F(T) = \frac{1}{5}c_1b_3T^5 + \frac{1}{4}(c_0b_3 + c_1b_2)T^4 + \frac{1}{3}(c_0b_2 + c_1b_1)T^3 + \frac{1}{2}(c_0b_1 + c_1b_0)T^2 + c_0b_0T \tag{26}$$

Equation (25) represents an exact solution in the case of a complete axial constraint of a simply supported column subject to a uniform temperature.

2.2.2. Case 2: Thermal buckling of a simply supported column with a linear-elastic constraint on axial displacements

As indicated above, the value of the buckling force for the column given by Eq. (18) is not affected by the history of thermal loading. The corresponding buckling strain is obtained by dividing this force by the cross-sectional area of the column and by the modulus of elasticity at the buckling temperature, i.e.,

$$\varepsilon_{cr} = \left(\frac{\pi}{\lambda}\right)^2 \tag{27}$$

Note that the buckling strain is not affected by the properties of the material, but only by the slenderness ratio, i.e., by geometry of the column.

The prebuckling axial displacement and strain are affected by the loading history. The prebuckling axial displacement of the column is approximated by

$$u(x,T) = V(T)\left(1 - 2\frac{x}{L}\right) \tag{28}$$

The displacement represented by Eq. (28) reflects the fact that maximum displacements occur at the elastically constrained ends of the column, while the displacement at the mid-span is equal to zero due to symmetry.

The following solution is possible in the incremental form dividing the range of temperature into intervals and gradually increasing temperature from the room value. Then, for the interval $T_i < T < T_{i+1}$, we know the values of displacements, strains, and stresses at $T = T_i$ from the analysis in the previous $i - th$ temperature interval. We can find the corresponding incremental additions to displacements, strains, and stresses at temperature $T = T_{i+1}$ using the linear incremental version of the boundary conditions (9) as follows.

Assuming a linear variation of displacements within the interval $T_i < T < T_{i+1}$, the incremental boundary condition at $x = 0$ can be written in the form

$$\begin{gathered} E(T_{i+1})A[\varepsilon(x=0,T_{i+1}) - \alpha(T_{i+1})T_{i+1}] - E(T_i)A[\varepsilon(x=0,T_i) - \alpha(T_i)T_i] = \\ k(T_{i+1})V(T_{i+1}) - k(T_i)V(T_i) \end{gathered} \tag{29}$$

All variables at temperature T_i in Eq. (29) are known from the solution at the previous temperature interval. Furthermore, $E(T_{i+1})$ and $\alpha(T_{i+1})$ are available from the corresponding polynomial property-temperature relationships (e.g., Eqs. (2) and (3)). Accordingly, the unknowns are only the strain $\varepsilon(x = 0, T_{i+1})$ and the amplitude of displacement $V(T_{i+1})$. However, at $x = 0$

$$\varepsilon(x=0,T_{i+1}) = -\frac{2}{L}V(T_{i+1}) \tag{30}$$

Accordingly,

$$V(T_{i+1}) = \frac{-E(T_{i+1})A\alpha(T_{i+1})T_{i+1} - E(T_i)A[\varepsilon(x=0,T_i) - \alpha(T_i)T_i] + k(T_i)V(T_i)}{\frac{2}{L}E(T_{i+1})A + k(T_{i+1})} \tag{31}$$

The strain and the amplitude of axial displacement at T_{i+1} can be calculated using Eqs. (30) and (31). Buckling occurs if the thermal strain becomes equal to the buckling strain given by Eq. (27).

2.2.3. Case 3: Thermal buckling of a simply supported column with a complete constraint on axial displacements—expansion to columns manufactured from arbitrary materials

As explained above, polynomials can represent relationships between material constants and temperature for every material, as long as the changes of properties occur continuously as temperature increases. In the case where the corresponding polynomials are different from those in Eqs. (2) and (3), the analysis neglecting the history of thermal loading is not affected, i.e., the previous approach remains valid. The generalization of the thermal buckling solution accounting for the history of thermal loading to the case of an arbitrary material is demonstrated below.

Consider a general case where the modulus of elasticity and the coefficient of thermal expansion are presented by the following polynomials:

$$E(T) = E\sum_{i=0}^{n-1} b_i T^i \tag{32}$$

and

$$\alpha(T) = \sum_{i=0}^{n-1} c_i T^i \tag{33}$$

Note that while the polynomials in Eqs. (32) and (33) contain the same number of terms, one of them, with a lower number of terms, can be simplified by setting the corresponding coefficients equal to zero. For example, in Eqs. (2) and (3), $n-1=3,$ and $c_2 = c_3 = 0$.

The buckling force for the column is still available by Eq. (18), i.e., the history of loading is not reflected in the "real-time" value of the modulus of elasticity. The thermal force should be found from Eq. (24) that requires a multiplication of two polynomials, (32) and (33). The product of these two polynomials is

$$E(T)\alpha(T) = E\sum_{s=0}^{s=2n-2} n_s T^s \tag{34}$$

where

$$n_j = \sum_{k=0}^{j} b_k c_{j-k} \tag{35}$$

Accordingly, the thermally-induced force in the column is

$$N(T)^T = AE\sum\nolimits_{s=0}^{2n-2} \frac{1}{s+1} n_s T^{s+1} \tag{36}$$

Equating the forces in Eqs. (18) and (36), the slenderness-buckling temperature relationship is

$$\lambda^2 = \frac{\pi^2 E(T_{cr})}{E} \frac{1}{\sum_{s=0}^{2n-2} \frac{1}{s+1} n_s {T_{cr}}^{s+1}} \tag{37}$$

In the case of steels 16Mn and 20MnTiB, Eq. (37) is reduced to Eq. (25).

3. Numerical analysis

Buckling and postbuckling characteristics of representative columns manufactured from 16Mn steel and 20MnTiB steel are illustrated in Figures 3–5. Relationships between the slenderness ratio of the column and its buckling temperature are depicted in Figure 3 for $\gamma = 0,\ 5,\ \mathit{and}\ 10,$ without accounting for the history of thermal loading. The values of γ are prescribed assuming that the stiffness of the boundary structures constraining axial displacements of the column vary with temperature in the same proportion as the modulus of elasticity of the column material. As indicated above, such assumption is sensible if the column and its supports are manufactured from the same material and experience identical temperatures. The case $\gamma = 0$ corresponds to a complete constraint against axial displacement, while two other cases are for an elastically constrained column. Note that the slenderness-buckling temperature relationships calculated by Eq. (20) coincide for both steels since their coefficients of thermal expansion vary with temperature according to the same law (see Table 1). One can observe in Figure 3 that in the presence of a stiffer constraint (smaller γ) buckling occurs at a lower temperature. Such observation is in an agreement with both experimental and theoretical studies [16, 17]. A slender column (with smaller γ) experiences thermal buckling at a lower temperature. Note that the columns characterized in Figure 3 remain elastic until bucking.

A perplexing observation from Figure 3 is that temperature corresponding to buckling of flexible columns may become quite low. This phenomenon occurs at different slenderness ratios, dependent on a degree of the axial elastic constraint. The same observations can be made from the solutions for thermal buckling of columns and plates obtained disregarding the effect of

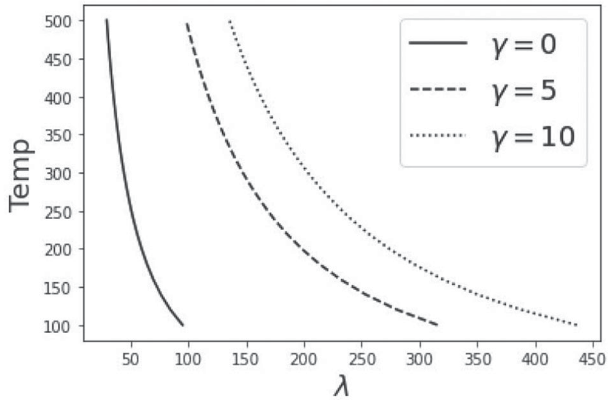

Figure 3. Relationship between the column slenderness ratio and buckling temperature. The history of thermal loading is neglected.

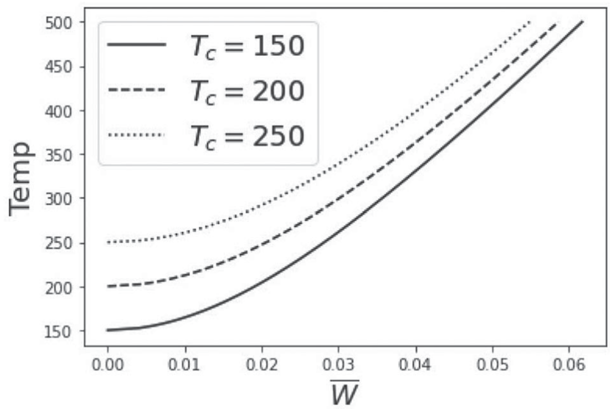

Figure 4. Postbuckling response of column upon thermal buckling. Temperature in degrees Celsius and the nondimensional deflection are shown along the vertical and horizontal axes, respectively. The history of thermal loading is neglected.

temperature on material properties (e.g., Jones, R.M. Buckling of Bars, Plates, and Shells. Bull Ridge Publishing, Blacksburg, Virginia, 2006). This implies that the structure buckles at a low temperature, but being axially constrained, it does not collapse, exhibiting instead a stable postbuckling response until the material becomes plastic.

Postbuckling deflections of completely constrained columns evaluated by Eq. (19) for various values of the buckling temperature are shown in Figure 4. Here, the buckling temperature is associated with the corresponding slenderness ratio as per Eq. (20). The results for two steels are identical demonstrating a stable postbuckling behavior. Such stable behavior is totally different from the response of mechanically compressed columns that collapse at even miniscule increments of the compressive force over the critical value. This can be explained by a difference in the boundary conditions in the axial direction. While the boundaries of mechanically compressed columns are unconstrained in the axial direction, in case of thermally loaded, axially constrained columns, fully or elastically constrained boundaries generate an axial tensile force once the column buckles, limiting deformations and reducing compressive stresses. The source of such tensile force is easy to comprehend since the length of a buckled column increases compared to the same column prior to buckling. Thus, the ends of the column trend to move toward each other and such trend is opposed by the constrained boundaries generating a tensile reactive force. In the contrary, in the case of mechanical compression of columns without an axial constraint at the

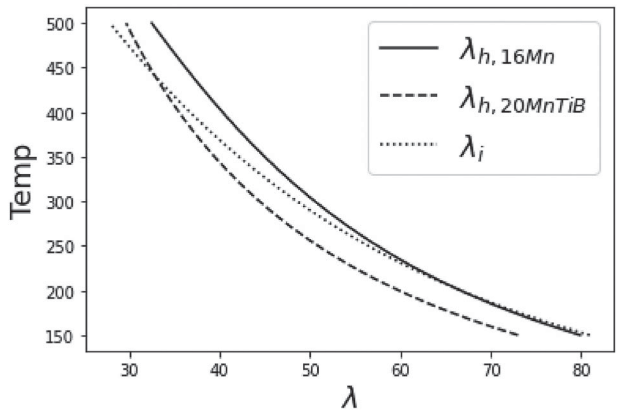

Figure 5. Relationships between the slenderness ratio and the buckling temperature of completely constrained in the axial direction columns neglecting (λ_i) and accounting for the history of thermal loading (λ_h).

boundaries, reactive forces do not appear since the boundaries simply move toward each other as the column buckles and there is not a "counterweight" to the effect of compression. At large values of the nondimensional deflection $\overline{W}$, sections of the column become plastic, eventually leading to its collapse. Thus, the right sections of the curves in Figure 4, exceeding $\overline{W}$ of an order 0.02 or 0.03 may be irrelevant.

The effect of the history of thermal loading on buckling temperatures of completely axially-constrained columns manufactured from steels 16Mn and 20MnTiB is shown in Figure 5 for the range of buckling temperatures from $T_{cr} = 150^0C$ to $T_{cr} = 500^0C$. The values of the slenderness ratio corresponding to the buckling temperatures shown in Figure 5 were calculated using Eqs. (20) and (25) where the former equation corresponds to using "real time" values of material constants, without accounting for the history of loading (λ_i), and the latter equation accounts for the history of loading (λ_h). While there is no difference between the curves for two steels when the history of loading is neglected, such difference is present if the history is accounted for. While the trends in the slenderness ratio-buckling temperature curves are the same in all cases shown in Figure 5, numerical differences are evident. The largest relative difference between the buckling temperatures obtained neglecting and accounting for the history of loading is observed for steel 20MnTiB at the slenderness ratio about 70, being of an order of 20%. Such difference justifies accounting for the history of thermal loading in the analysis. Note that buckling temperatures calculated accounting for the history of thermal loading are generally higher for steel 16Mn compared to the result obtained neglecting this history, except for large slenderness ratios. However, for steel 20MnTiB, the comparison yields an opposite observation, i.e., buckling temperatures found accounting for the history of thermal loading are lower than those neglecting this history, except for columns with a small slenderness ratio. Therefore, it is impossible to predict in advance whether neglecting the history of thermal loading is conservative or unsafe.

4. Conclusions

a. There are significant differences between buckling temperatures found neglecting the history of loading as compared to those accounting for it. It is impossible to predict in advance which approach yields more conservative results.
b. There is a difference between postbuckling temperature-deflection response of heated columns and postbuckling response of mechanically compressed columns. The latter problem is characterized by a nearly neutral equilibrium under compression beyond the critical force. Accordingly, even minor increments of the force over the buckling value cause catastrophic

deflections. In the contrary, in the case of a thermal postbuckling, the columns exhibit a significant postbuckling life, i.e., they can endure large temperature increments, even after buckling. This difference may be explained by different boundary conditions in the axial direction. The boundaries of mechanically compressed columns must be free in the axial direction, so that they cannot apply a reactive force to a buckled column. In the contrary, thermally loaded columns should be either fully or elastically constrained against axial displacements to generate thermal compressive stresses that cause buckling. Upon thermal buckling, the boundaries generate reactive tensile forces that reduce the effect of compressive stresses produced by heating.

c. A larger axial constraint at the boundaries of a column results in a lower buckling temperature. This is because of larger axial displacements at the same temperature in a column with more compliant boundaries. As a result, higher temperature is needed to develop buckling force in such columns. In the contrary, the axial displacements of the column are reduced if the boundaries are stiffer in the axial direction of the column (i.e., if the axial constraint at the boundaries is larger), so that buckling force can be reached at a lower temperature.

d. Considering that material properties are continuous functions of temperature, according to the Weierstrass Theorem, they can always be presented by polynomials that are similar to Eqs. (1)–(3), though the coefficients and the number of terms in such polynomials depend on the material. Accordingly, the methods used to characterize thermal buckling and postbuckling can be applied to columns manufactured from practically any material, as long as the above-mentioned polynomials are available from experimental data.

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