Lecture 16: Gravitational potential energy and space travel

- Universal gravitational potential energy
- Space travel problems
- Escape speed
- Orbital energy
- Multiple objects

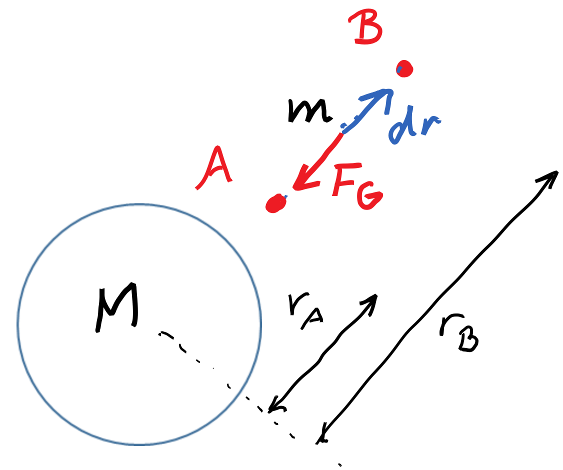
Gravitational potential energy

$F_{grav} = \frac{GmM}{r^2}$, attractive

Conservative force

$$U_B - U_A = -W_{A\rightarrow B} = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r}$$

$$U_B - U_A = -W_{A\to B} = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{F}\cdot d\vec{r}$$



$$\int_{\vec{r}_A}^{\vec{r}_B} \vec{F}\cdot d\vec{r} = \int_{r_A}^{r_B} F_r dr = \int_{r_A}^{r_B} -\frac{GmM}{r^2} dr$$

$$= -\left[-\frac{GmM}{r}\right]_{r_A}^{r_B} = \frac{GmM}{r_B} - \frac{GmM}{r_A}$$

$$U_B - U_A = -GmM\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

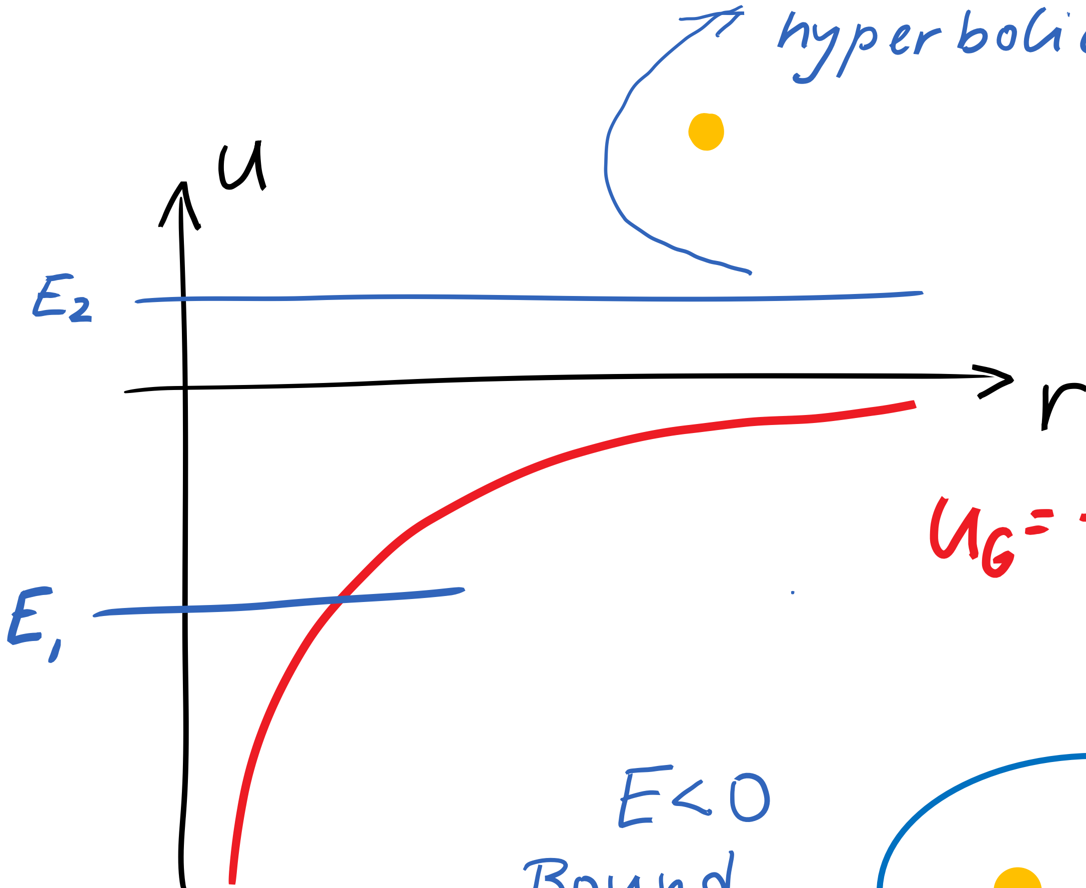
Choose reference point: $r_0 = \infty$.
Assign $U(r_0 = \infty) = 0$

$$\boxed{U_{Grav} = -\frac{GmM}{r}}$$

negative!

Potential energy diagram

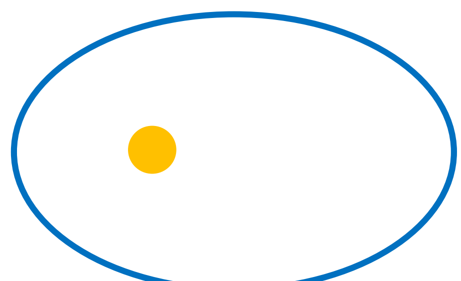
hyperbolic trajectory



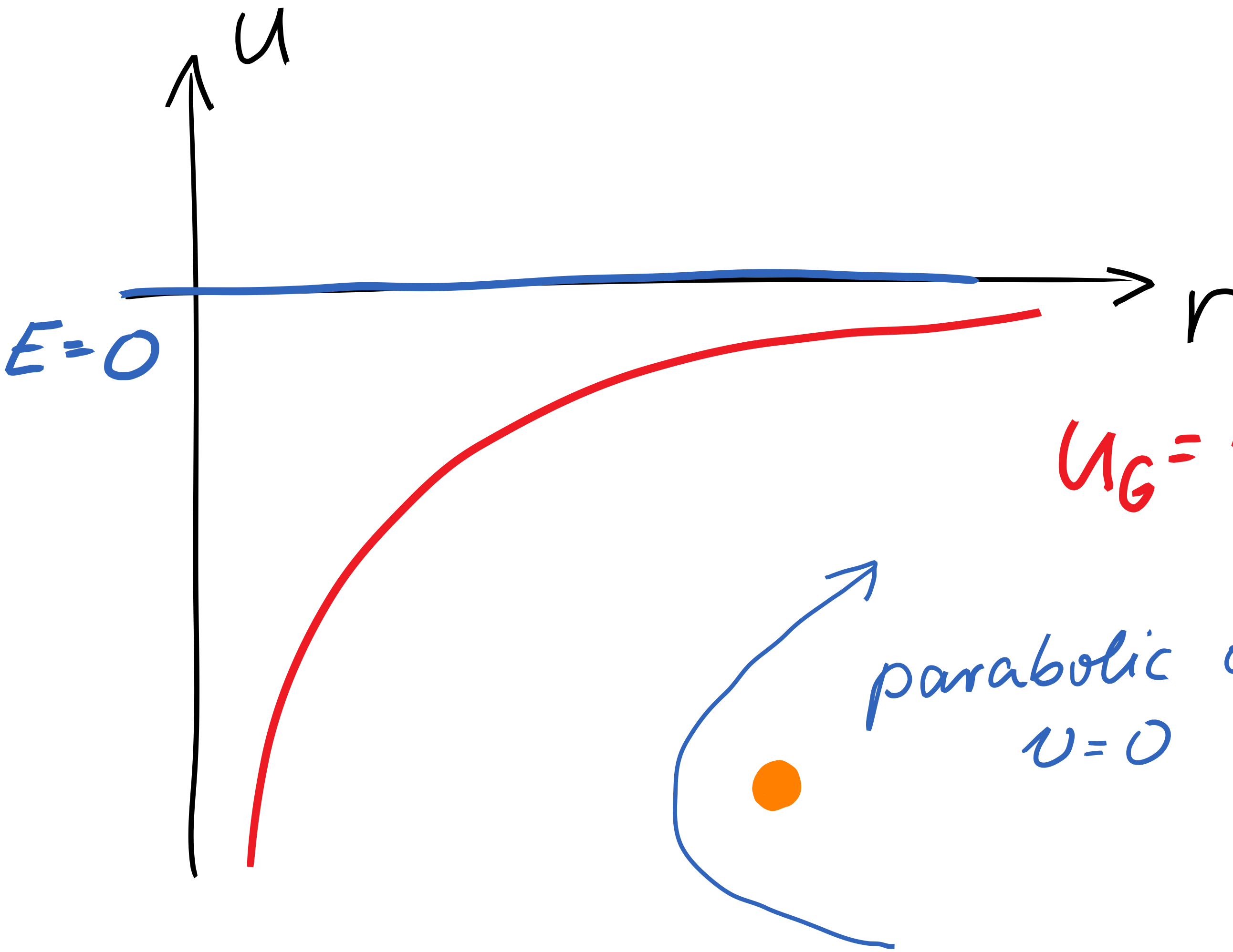
$$U(r_0 = \infty) = 0$$

$U_G = -\frac{GMm}{r}$

E<0
Bound orbits



Potential energy diagram

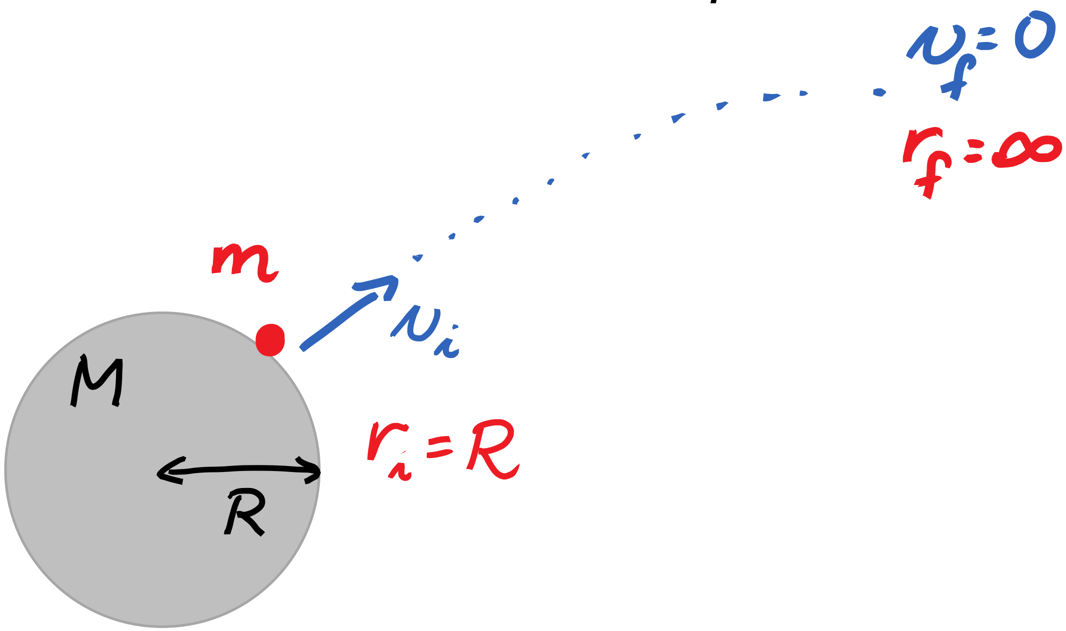


$$U_G = -\frac{GMm}{r}$$

parabolic orbit
$v = 0$ @ $r = \infty$

Escape condition

$$E = \tfrac{1}{2}mv^2 - \frac{GmM}{r}$$



Critical escape condition: *barely* making it to $r = infinity$, having slowed to speed of zero

Escape speed

Minimum speed an object must have at distance *R* from central mass *M* if it is to go infinitely far away

$$E_i = E_f$$

$$½mv_{esc}^2 - \frac{GmM}{R} = \cancelto{0}{½mv_\infty^2} - \cancelto{0}{\frac{GmM}{r=\infty}}$$

$$½\cancel{m}v_{esc}^2 - \frac{G\cancel{m}M}{R} = 0 - 0$$

$$\boxed{v_{esc} = \sqrt{\frac{2GM}{R}}}$$

Example: Escape speed from Earth

$$M_{Earth} = 5.97 \times 10^{24} kg$$
$$R_{Earth} = 6.38 \times 10^{6}\, m$$

$$G = 6.67 \times 10^{-11}\, N\, m^2/kg^2$$

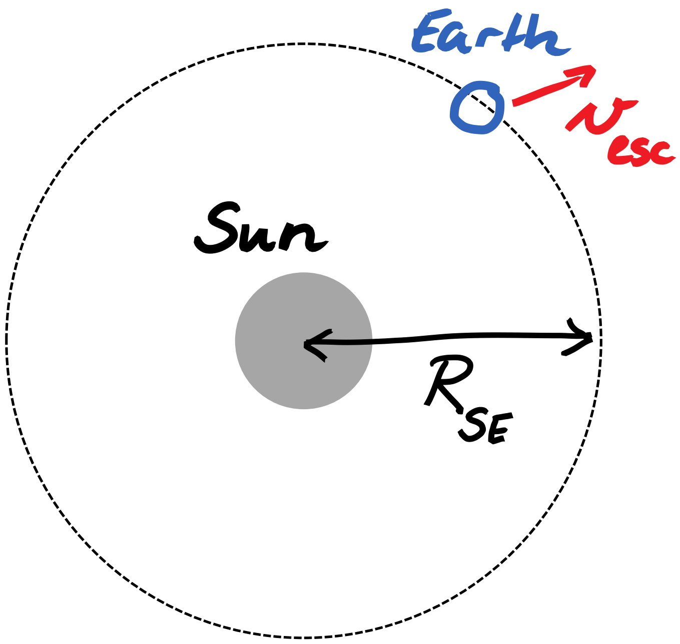
$$v_{esc} = \sqrt{\frac{2GM}{R}} = 11{,}200\, m/s$$

Example: Escape speed from orbit

Speed necessary to escape gravitational field of the Sun when object is launched from Earth:

M_{Sun}=1.99 × 10 ^{30} kg
R_{SE}= 1.50 × 10 ^{11} m
G= 6.67 × 10 $^{-11}$ N m^2/kg^2

$$v_{esc} = \sqrt{\frac{2GM_{Sun}}{R_{SE}}} = 42{,}000\ m/s$$



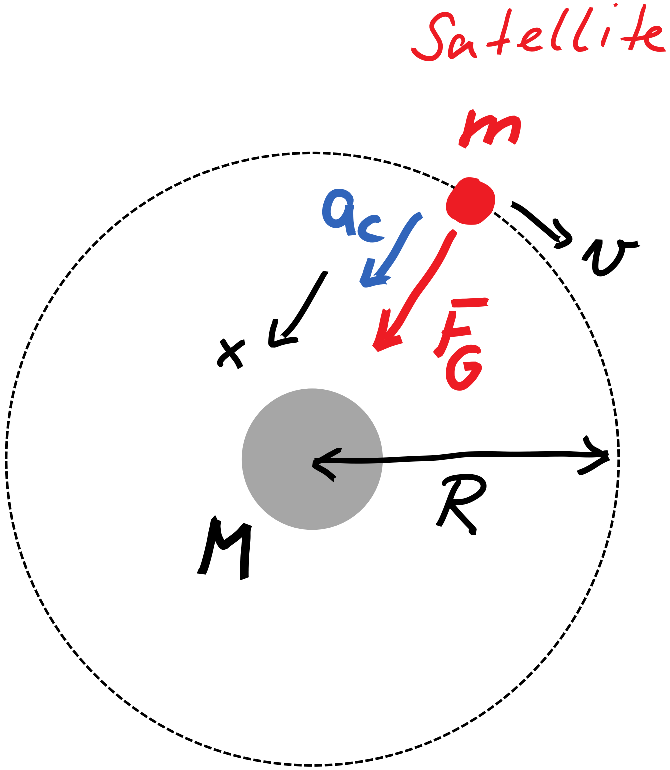
Orbital Energy

$$E = K + U = \tfrac{1}{2}mv^2 - \frac{GmM}{R}$$

Speed of satellite in circular orbit: $v^2 = \frac{GM}{R}$

$$E = \tfrac{1}{2}m\left(\frac{GM}{R}\right) - \frac{GmM}{R}$$

$$\boxed{E = -\frac{GmM}{2R}}$$

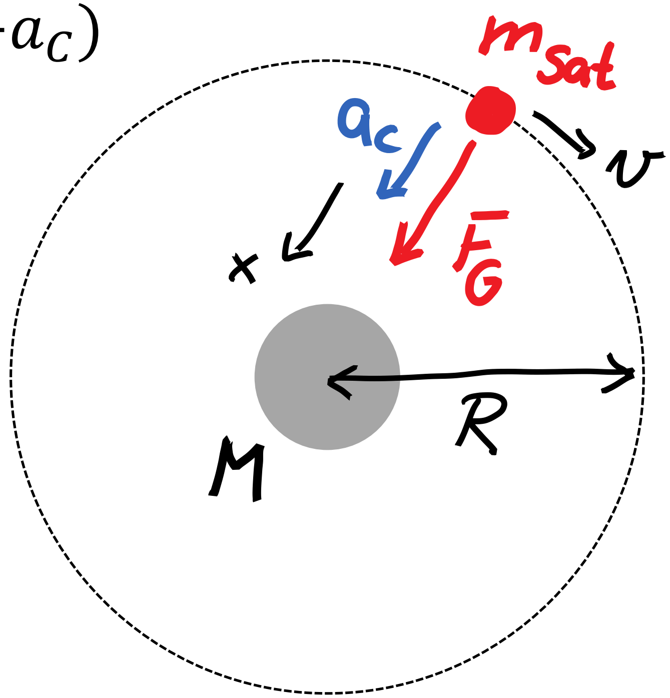


Satellite Motion

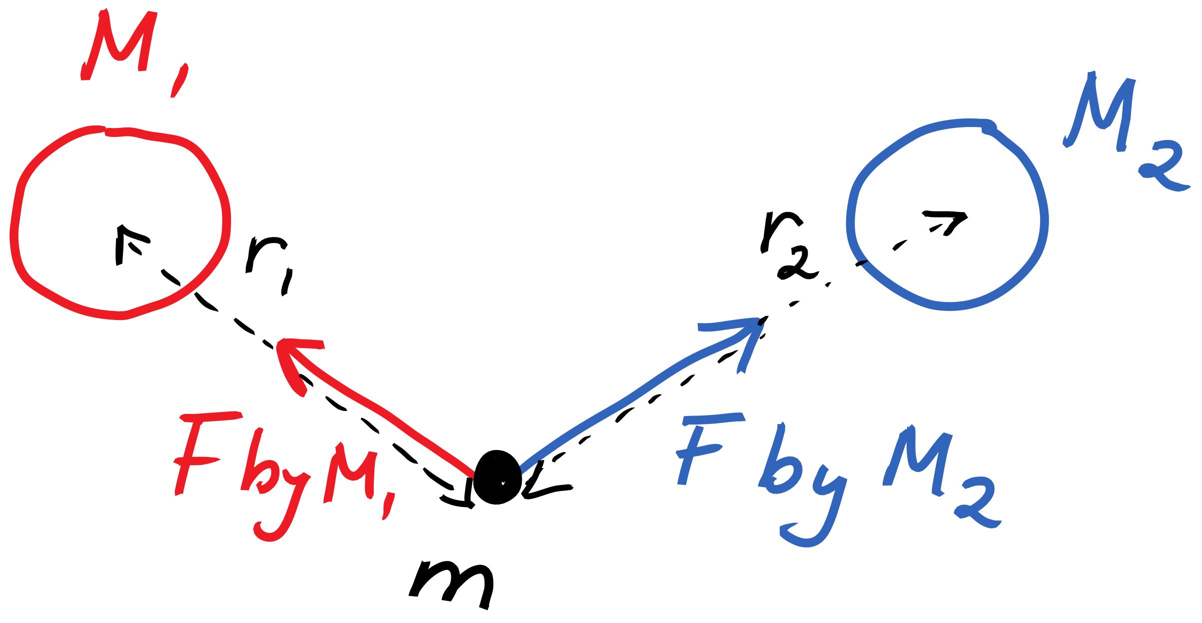
$$\sum F_x = F_{grav,x} = m_{sat}a_x = m_{sat}(+a_C)$$

$$\frac{Gm_{sat}M}{R^2} = m_{sat}\frac{v^2}{R}$$

$$\frac{GM}{R} = v^2$$



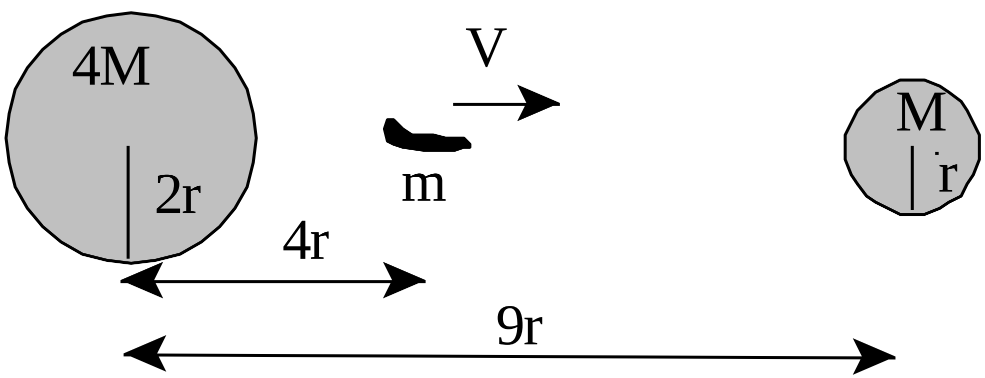
Multiple objects



$$U_g = U_{g,1} + U_{g\,2} = -\frac{GM_1 m}{r_1} + \left(-\frac{GM_2 m}{r_2}\right)$$

Example with multiple objects

A planet has mass $4M$ and a radius $2r$. Its moon has mass M and radius r. The centers of the planet and moon are a distance $9r$ apart.
A shuttle of mass m is a distance $4r$ away from the center of the planet and moving with speed V.
What is the total mechanical energy of the shuttle?



If the shuttle was initially at rest at X, how much work did the engines do?