Lecture 23:
Angular momentum

- Angular momentum of a point mass
- Angular momentum of a rigid rotating object
- Conservation of angular momentum

Translation vs rotation

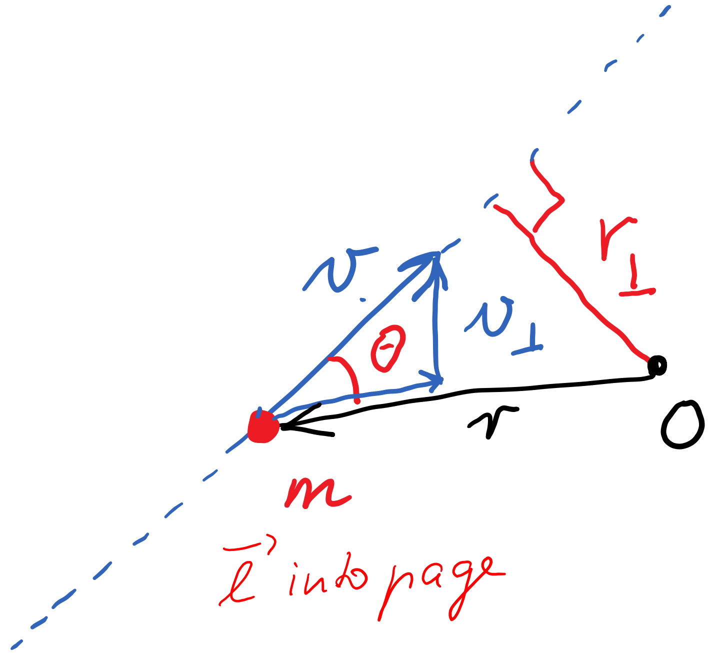
Linear momentum $\vec{p}$ is fundamental quantity for translation.
Forces change linear momentum.

Angular momentum $\vec{l}$ is fundamental quantity for rotation.
Torques change angular momentum.

$$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

Angular momentum of a particle

$$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$



$$l = r_{\perp} m v = r m v_{\perp} = r m v \sin\theta$$

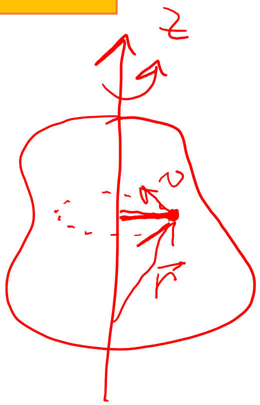
Direction: right hand rule

$\vec{r} \times \vec{p} = \vec{l}$

thumb — index finger — middle finger

perpendicular to plane of $\vec{r}, \vec{p}$

Angular momentum of rigid object



For system of particles: $\vec{L} = \sum_n \vec{l}_n$

$$\sum_n l_n = \sum_n r_{n\perp} m_n v_n = \sum_n r_{n\perp} m_n (\omega_n \, r_{n\perp}) = \omega \sum_n m_n {r_{n\perp}}^2$$

$$\vec{L} = I\vec{\omega}$$

$\vec{L}$ is in the same direction as angular velocity $\vec{\omega}$ vector for rotations **about a symmetry axis only**.

This is not the case for rotation about other axes.

Angular momentum conservation

$$\sum \tau_z = \frac{dL_z}{dt} = \frac{d(I\omega_z)}{dt} = I\alpha_z$$

For system: $$\sum \vec{\tau} = \sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

If $$\sum \vec{\tau}_{ext} = 0 \Longrightarrow \frac{d\vec{L}}{dt} = 0, \vec{L}_i = \vec{L}_f$$

Compare: $\sum F_x = \frac{dP_x}{dt}$ If $\sum F_{ext,x} = 0, P_{ix} = P_{fx}$

Demonstrations

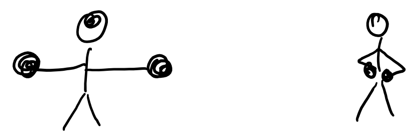
$$\sum \cancel{\vec{\tau}_{ext}}^{\,0} = \frac{d\vec{L}}{dt}$$

$$L_{iz} = L_{fz}$$

$$I_i \, \omega_i = I_f \, \omega_f$$

$$L_z = I \omega_z$$

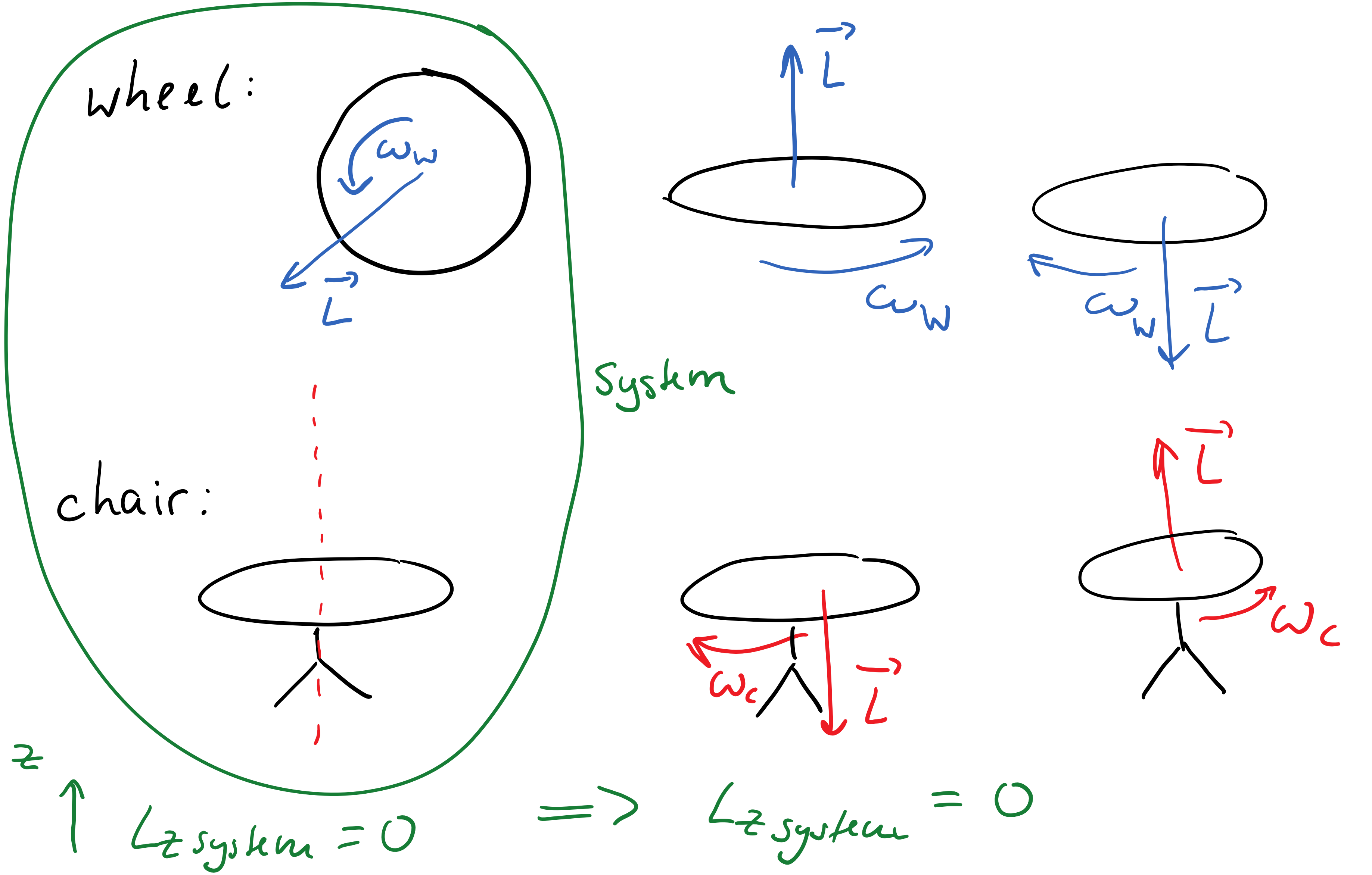
$$I = \sum_n m_n r_n^2$$



big I → small ω

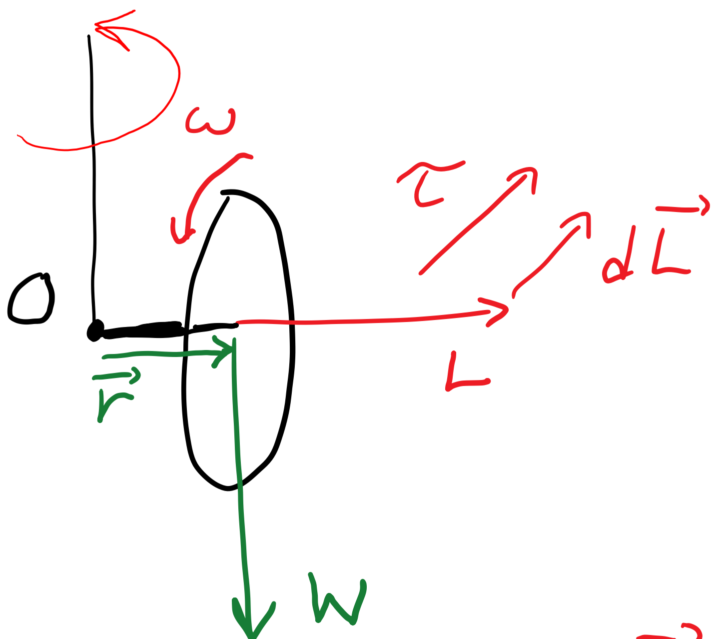
small I → large ω

Demonstrations



Demonstrations

Bicycle wheel gyroscope



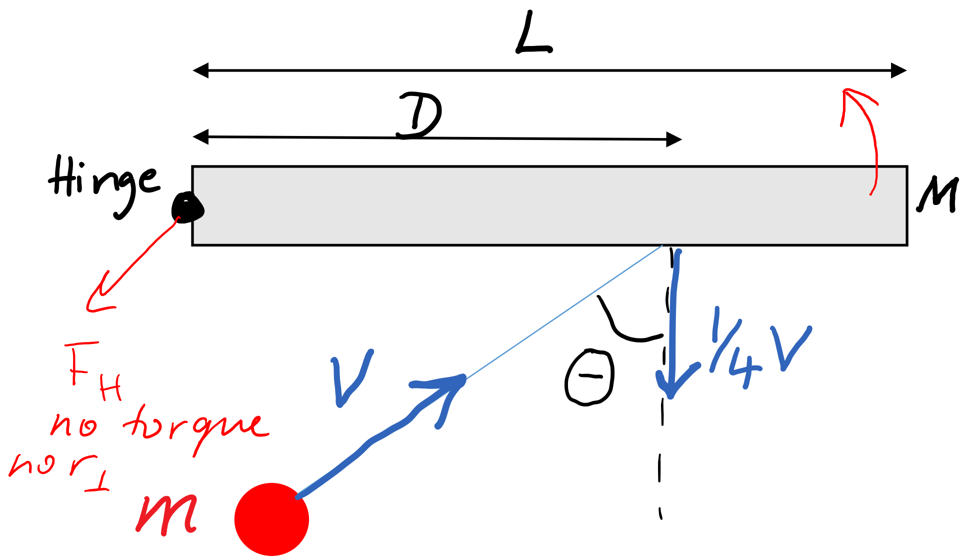
$\vec{\tau}_{weight} \perp \vec{r}, \vec{W}$ into page

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$d\vec{L} = \vec{\tau}\, dt$$

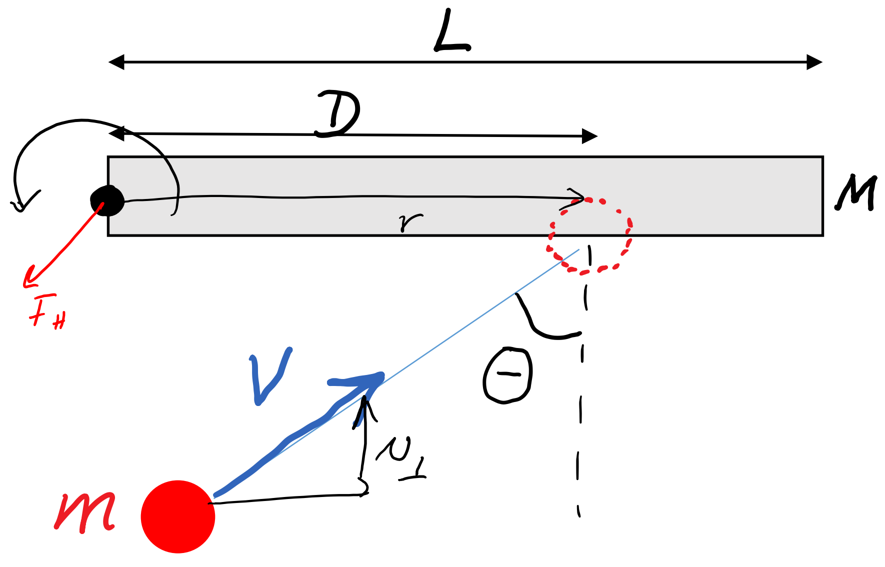
Example 1

A ball of mass m and speed V strikes a door at angle θ and bounces off at a right angle with ¼ its original speed. What is the final angular speed of the door after the collision?



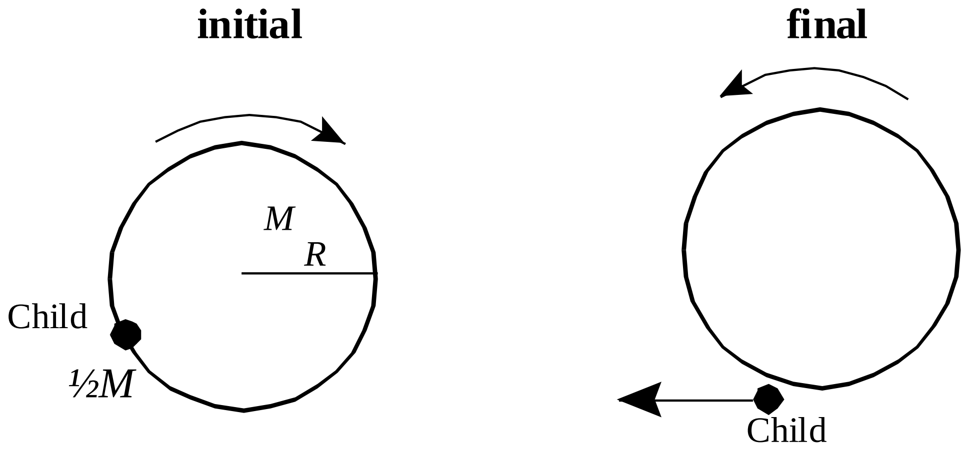
Example 2

A ball from example 1 is made of putty and sticks to the door after the collision. What is the final angular speed of the door with the ball stuck on?



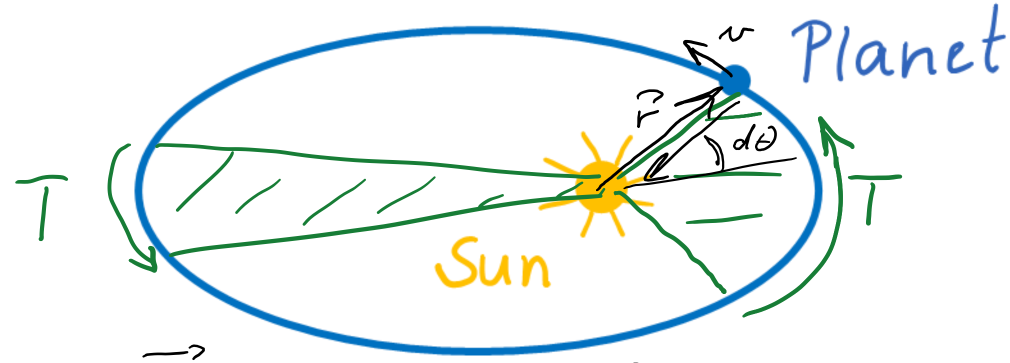
Example 3

A merry-go-round (solid disk of mass M and radius R) is rotating on frictionless bearings about a vertical axis through its center. It rotates in the clockwise direction with angular speed ω. A child of mass $\frac{1}{2}M$ is initially sitting at the outer edge of the merry-go-round. When the child jumps off tangentially to the circumference, the merry-go-round reverses its rotation and now rotates with the same angular speed ω in the opposite, i.e. counterclockwise, direction. Derive an expression for the speed relative to the ground with which the child jumps off.



Kepler's 2nd Law

A line drawn between the sun and a planet sweeps out equal areas in equal intervals of time

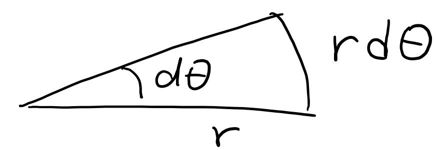


Torque by gravity: $\vec{\tau} = \vec{r} \times \vec{F}_{grav} = 0 \Rightarrow \vec{l} = \text{constant}$

$$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

$$l = rmv_{\perp} = rm\omega r = r^2 m\omega$$

Area: $dA = \frac{1}{2} r (r\, d\theta) = \frac{1}{2} r^2 d\theta$



$$\frac{dA}{dt} = \frac{1}{2} r^2 \underbrace{\frac{d\theta}{dt}}_{\omega} = \frac{1}{2} r^2 \omega = \frac{l}{2m} = \text{constant}$$