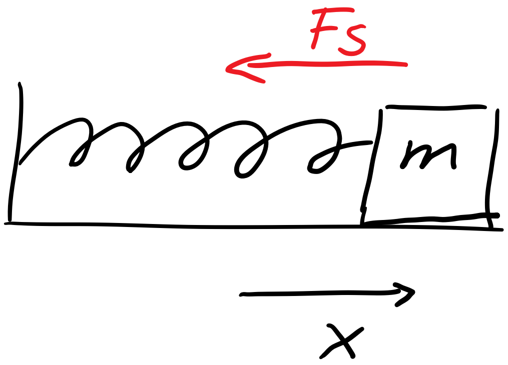
Lecture 24: Periodic Motion

- Motion of a mass at the end of a spring
- Differential equation for simple harmonic oscillation
- Amplitude, period, frequency and angular frequency
- Energetics
- Simple pendulum
- Physical pendulum

Mass at the end of a spring

Mass m connected to a spring with spring constant k on a frictionless surface



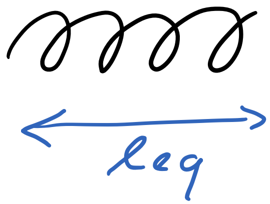
$$F_x = -kx$$

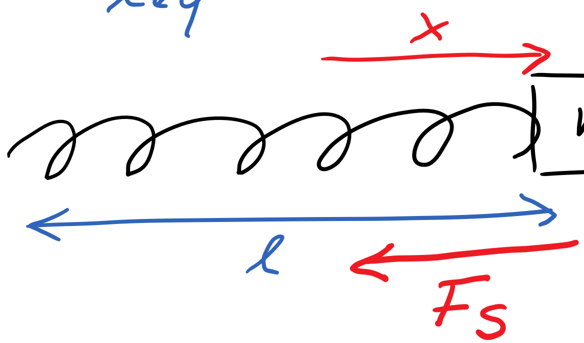
Linear restoring spring force

Spring force

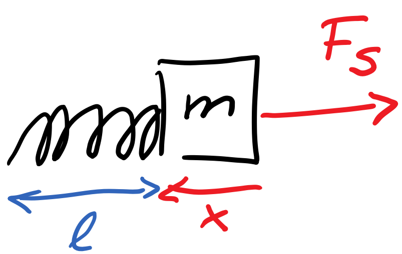
Spring force: $F_{S\,x} = -kx$

$x = l - l_{eq}$
stretch or compression
k force constant





F_x is negative if x is positive
(stretched spring)



F_x is positive if x is negative
(compressed spring)

Differential equation of a SHO

Newton's 2nd Law: $\sum F_x = ma_x$

$$-kx = m\frac{d^2x}{dt^2}$$

$$-\frac{k}{m}x = \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$ *

$$\omega = \sqrt{\frac{k}{m}}$$ Angular frequency

Differential equation of a Simple Harmonic Oscillator

*We can always write it like this because m and k are positive

Solution

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Equation for SHO

General solution: $x = A\cos(\omega t + \varphi)$

$$\frac{dx}{dt} = -A\omega \sin(\omega t + \varphi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \varphi) = -\omega^2 x$$

A and φ: two "constants of integration" from solution of a *second-order* differential equation.
Determined by the initial conditions.

Amplitude

$$x = A\cos(\omega t + \varphi)$$

Range of cosine function: -1…+1

$\Longrightarrow \quad -A \leq x(t) \leq +A$

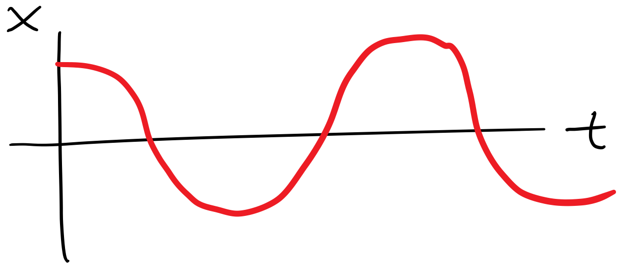
A = Amplitude of the oscillation

Phase Constant

$$x = A\cos(\omega t + \varphi)$$

If φ=0:

$$x = A\cos(\omega t)$$
$$x(t = 0) = x_0 = A$$

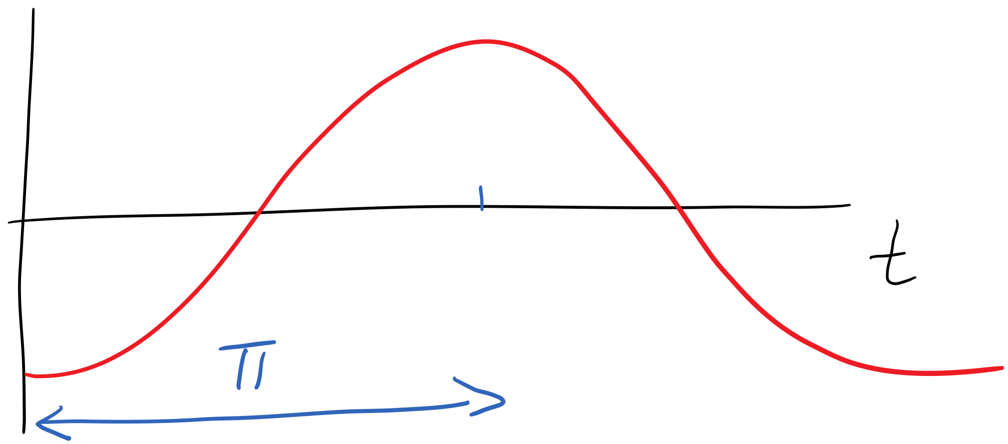


To describe motion with different starting points:
Add phase constant to shift the cosine function

$$x = A\cos(\omega t + \varphi)$$

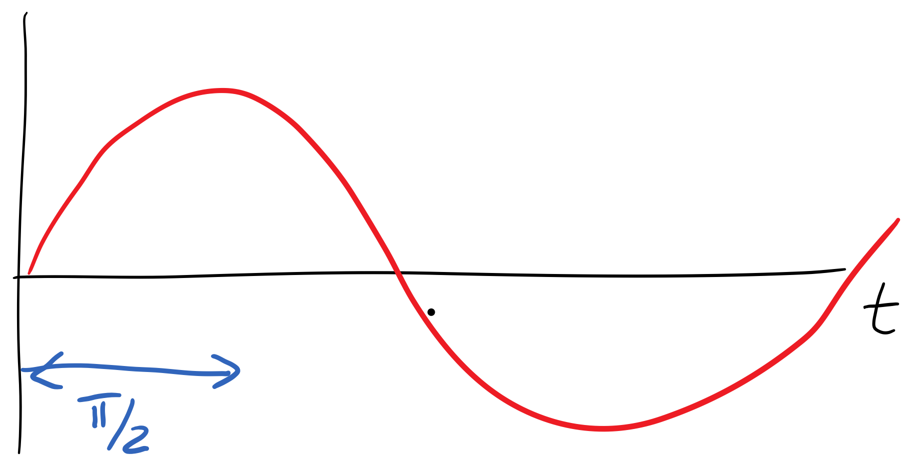
$x_0 = -A$:

shift by π



$x_0 = 0$:

shift by $\frac{\pi}{2}$



Initial conditions

$$x_0 = x(t = 0)$$
$$v_{x0} = v_x(t = 0)$$

$$x_0 = A\cos(0 + \varphi) = A\cos(\varphi)$$
$$v_{x0} = -A\omega\, sin(0 + \varphi) = -A\omega\, sin(\varphi)$$

$\rightarrow$ two equations for A and φ

Position and velocity

$$x = A\cos(\omega t + \varphi)$$

$$v_x = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$$

At time t_m: $x = x_{max} = A$ $\qquad \cos(\omega t_m + \varphi) = 1$

$$(\omega t_m + \varphi) = 0\ or\ \pi$$

$$\sin(\omega t_m + \varphi) = 0 \quad \Longrightarrow \quad v_x(t_m) = 0$$

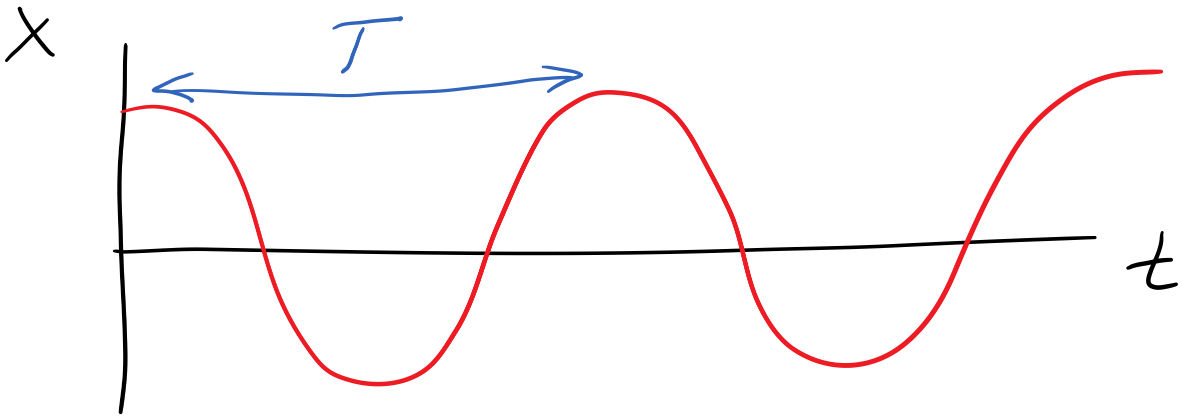
Mass stops and reverses direction when it reaches maximum displacement (turning point)

Simulation

http://www.walter-fendt.de/ph14e/springpendulum.htm

Period and angular frequency

$$x = A\cos(\omega t + \varphi)$$



Time T for one complete cycle: period

$(\omega t + \varphi)$ changes by 2π in time T

$$\omega T = 2\pi \quad \Longrightarrow \quad \omega = \frac{2\pi}{T} = 2\pi f$$

Effect of mass and amplitude on period

$$\omega T = 2\pi \implies T = \frac{2\pi}{\omega}$$

$$\omega = \sqrt{\frac{k}{m}} \qquad \implies T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Amplitude A does not appear – no effect on period

Demo: Vertical springs showing effect of m and A

Energy in SHO

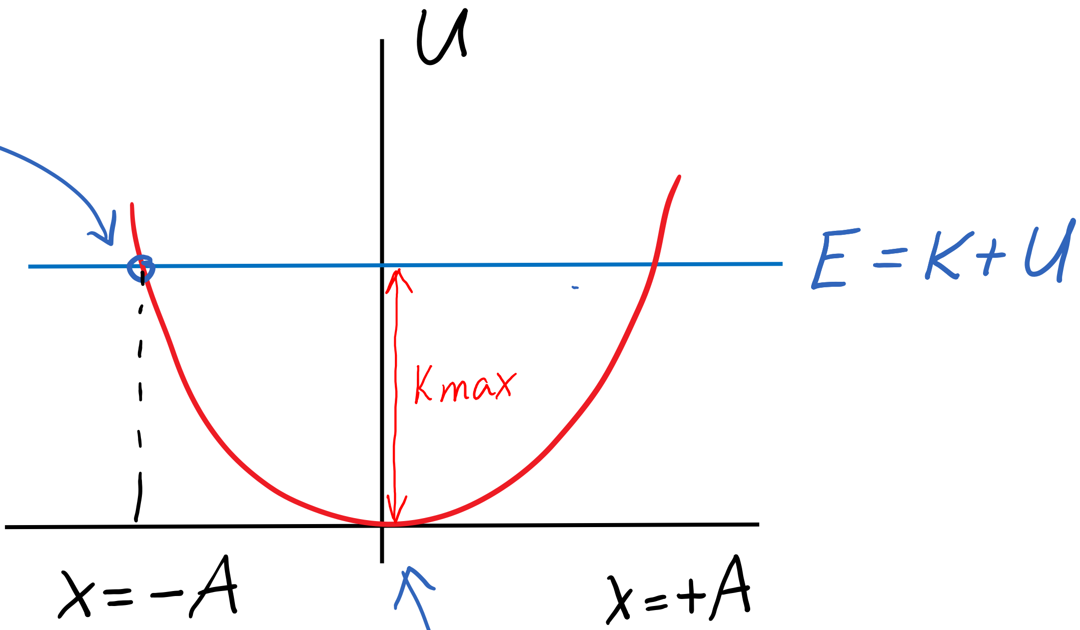
Potential energy of spring force: $U = \frac{1}{2}kx^2$

at $x = \pm A$:

$U = \frac{1}{2}kA^2$

$K = 0$

$E = ½kA^2$



at $x = 0$:

$U = 0$

$K = K_{max} = E$

Kinetic and potential energy in SHO

$$K = ½mv^2 = ½m[A\omega \sin(\omega t + \varphi)]^2$$

$$K_{max} = ½\, mv_{max}^2 = ½\, m\, (\omega A)^2$$

$$U = ½kx^2 = ½k[A \cos(\omega t + \varphi)]^2$$

$$U_{max} = ½kx_{max}^2 = ½\, k\, A^2$$

$$E = K_{max}\, sin^2\,(\omega t + \varphi) + U_{max} cos^2\,(\omega t + \varphi)$$

http://www.walter-fendt.de/ph14e/springpendulum.htm

Example

A block of mass M is attached to a spring and executes simple harmonic motion of amplitude A. At what displacement(s) x from equilibrium does its kinetic energy equal twice its potential energy?

SHO

$$\frac{d^2x}{dt^2} = -\,\omega^2 x$$

Equation for SHO

General solution:

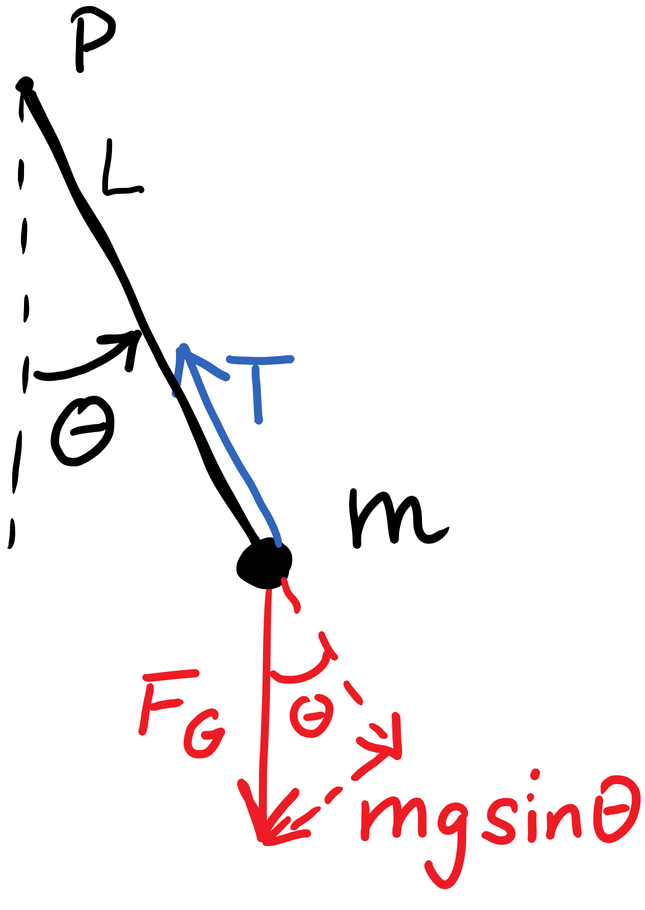
$$x = A\cos(\omega t + \varphi)$$

$$T = \frac{2\pi}{\omega}$$

Simple Pendulum

Point mass m at the end of a massless string of length L

θ **=** displacement coordinate (**with sign**) from vertical equilibrium position



Simple Pendulum

$$\Sigma\tau_z = I\alpha_z$$

$$-mg\,L\,sin\theta = mL^2\frac{d^2\theta}{dt^2}$$

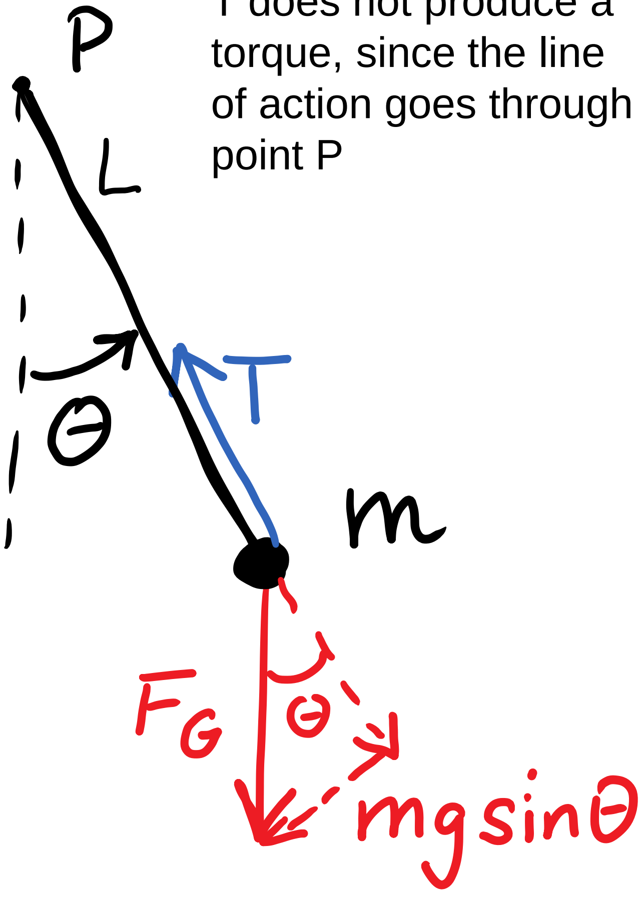
Very complicated differential equation!
But for small oscillations:

$$sin\theta \approx \theta$$

And

$$-\frac{g}{L}\theta = \frac{d^2\theta}{dt^2}$$

Differential equation of SHO



T does not produce a torque, since the line of action goes through point P

Simple pendulum oscillations

$$-\frac{g}{L}\theta = \frac{d^2\theta}{dt^2}$$

Differential equation of simple harmonic oscillator

$$\theta(t) = \theta_{max}\cos(\omega t + \varphi)$$

With $\omega = \sqrt{\frac{g}{L}}$ and

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Demo: Simple pendulum with different masses, lengths and amplitudes

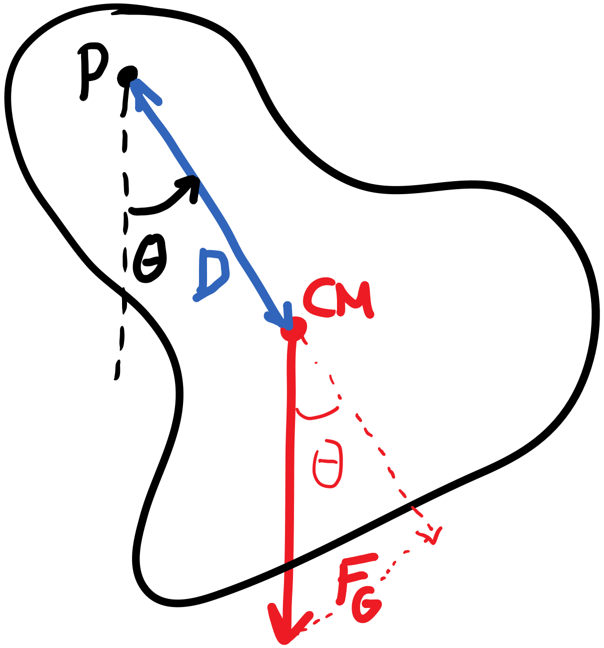
$$T = 2\pi\sqrt{\frac{L}{g}}$$

Demo: Simple pendulum with different masses, lengths and amplitudes

- Period independent of mass
- Period independent of amplitude

Physical Pendulum

Extended object of mass m that swings back and forth about an axis ***P*** that does not go through its center of mass **CM**.



$$\Sigma\tau_z = I\alpha_z$$

$$-mg\ D\ sin\theta = I\,\frac{d^2\theta}{dt^2}$$

For small oscillations: $sin\theta \approx \theta$

$$-\frac{mgD}{I}\theta = \frac{d^2\theta}{dt^2}$$

SHO

Motion of the Physical Pendulum

$$-\frac{mgD}{I}\theta = \frac{d^2\theta}{dt^2}$$

ω^2

SHO

$\theta = \theta_{max}\cos(\omega t + \varphi)$

$\omega = \sqrt{\frac{mgD}{I}}$

$$T = 2\pi\sqrt{\frac{I}{mgD}}$$

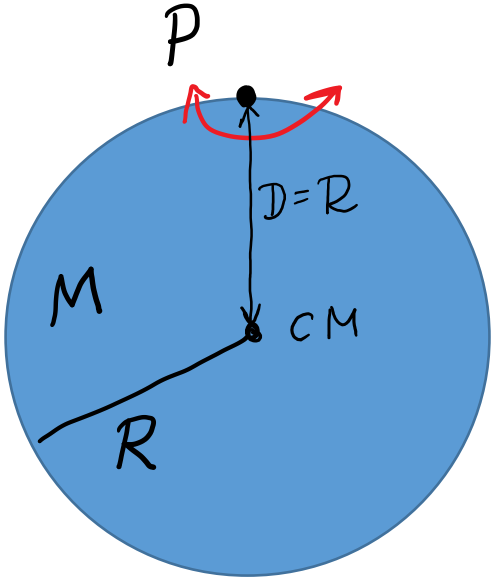
I is moment of inertia about axis P
D is distance between P and CM
Parallel axis theorem:

$$I_P = I_{CM} + mD^2$$

Demo: Meter stick pivoted at different positions

Example

A uniform disk of mass M and radius R is pivoted at a point at the rim. Find the period for small oscillations.



$$T = 2\pi\sqrt{\frac{I}{mgD}}$$

$$I_P = I_{CM} + mD^2$$