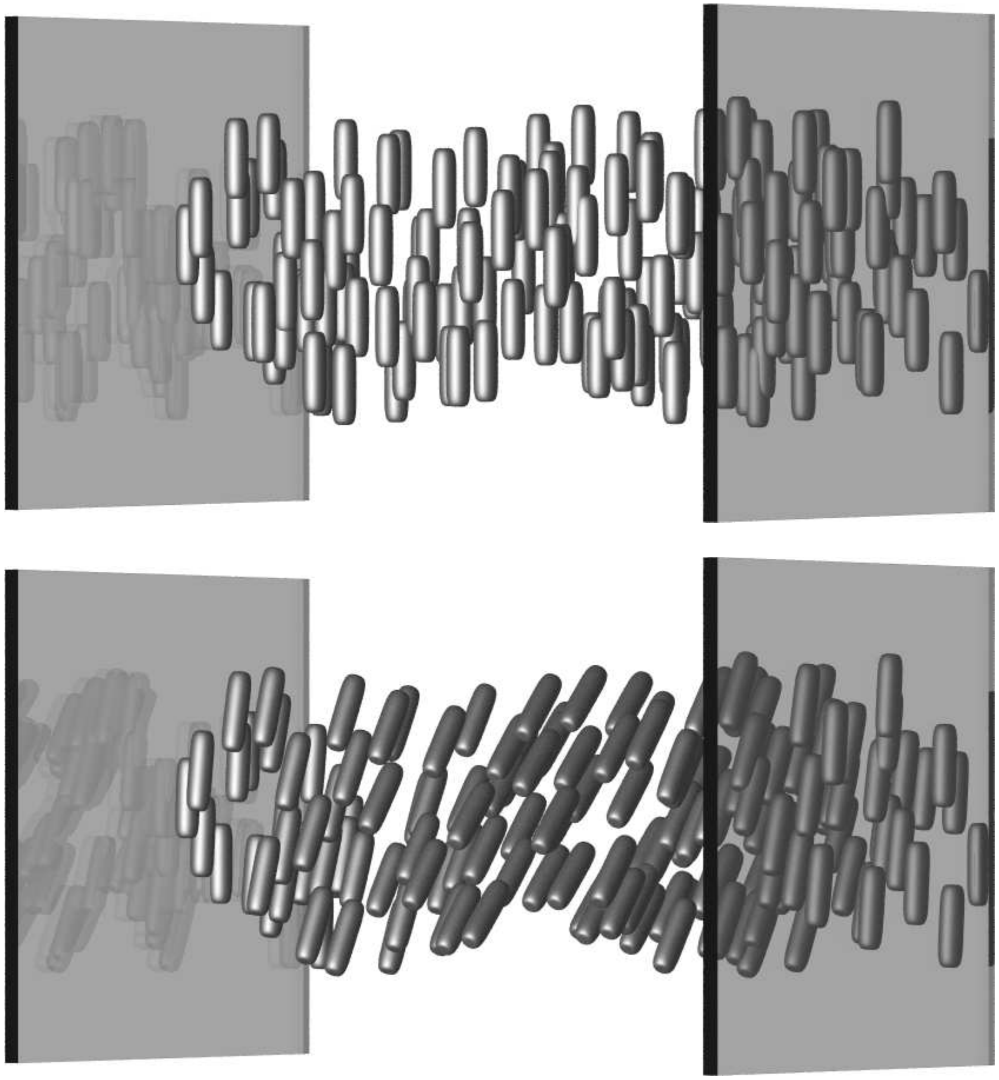
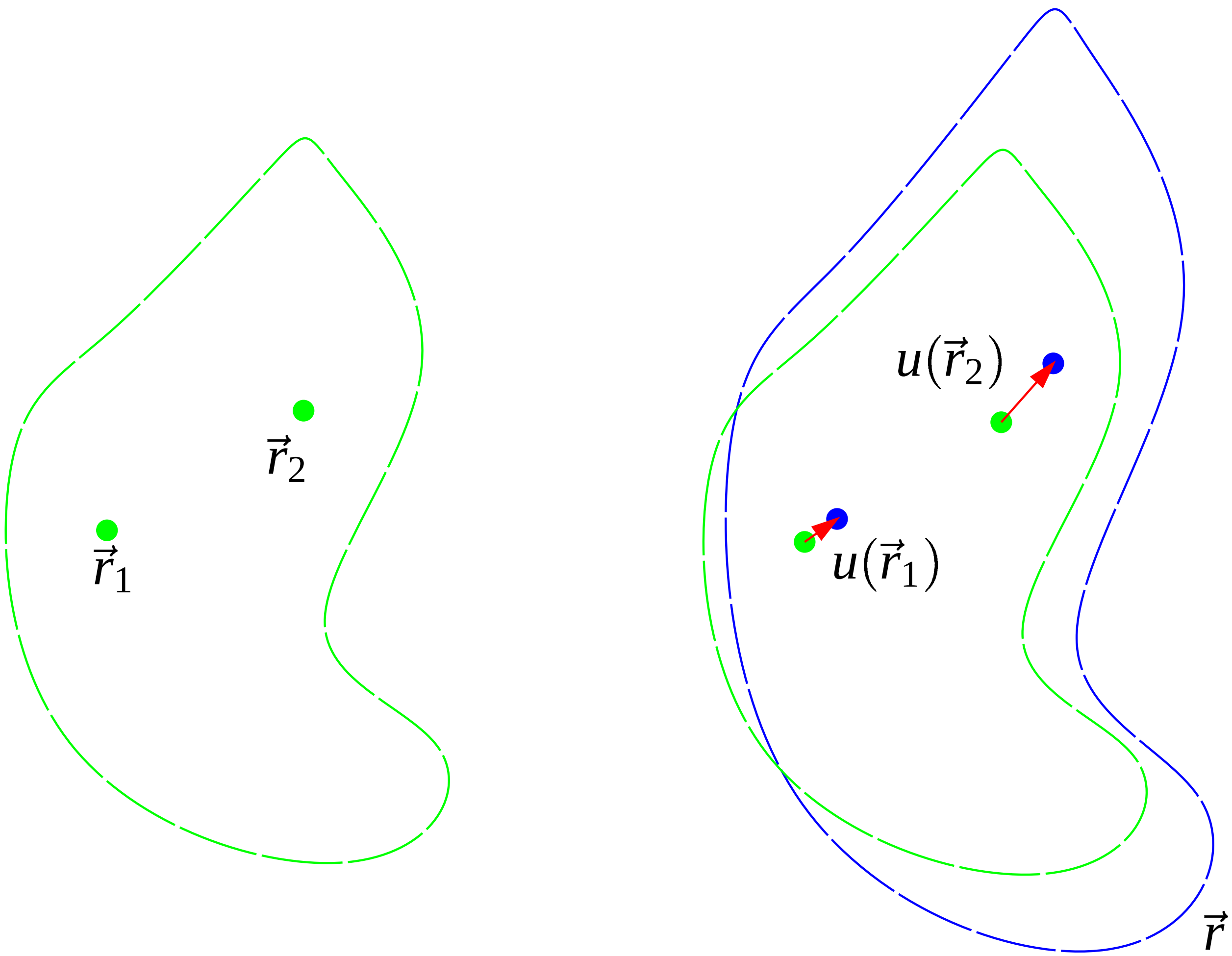
Elasticity



Before deformation After deformation



$$\vec{r} \qquad \vec{r}+\vec{u}(\vec{r}). \tag{L1}$$

Many ways to derive elasticity. Could derive from theory of atoms and their interactions. However, this approach is not historically accurate, and not fully general.

General Theory of Linear Elasticity

Most general approach modeled by Landau; construct free energy simply by considering symmetry and using fact that deformations are small:

- $\vec{u}$ vanishes in equilibrium
- Free energy invariant under translation.
- Smallest allowed powers or $\vec{u}$
- Derivatives of lowest allowed order
- Uniform rotation costs no energy.

Unique (?) free energy consistent with these constraints:

$$\mathcal{F} = \int d\vec{r}\, \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} E_{\alpha\beta\gamma\delta} \frac{\partial u_\alpha(\vec{r})}{\partial r_\beta} \frac{\partial u_\gamma(\vec{r})}{\partial r_\delta}. \tag{L2}$$

45 independent $E_{\alpha\beta\gamma\delta}$ after considering symmetry under interchange of indices.

$$u_\alpha = \phi \sum_{\beta\mu} \epsilon^{\alpha\beta\mu} r_\beta n_\mu. \tag{L3}$$

$$\sum_{\alpha\beta\gamma\delta\mu\mu'} \int d\vec{r}\, \epsilon^{\alpha\beta\mu} n_\mu E_{\alpha\beta\gamma\delta} \epsilon^{\gamma\delta\mu'} n_{\mu'} = 0 \tag{L4}$$

$$\Rightarrow \quad E_{\alpha\beta\gamma\delta} - E_{\beta\alpha\gamma\delta} - E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\delta\gamma} = 0. \tag{L5}$$

Strain tensor

Define strain tensor

$$e_{\alpha\beta} \equiv \tfrac{1}{2}\left[\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha}\right] \tag{L6}$$

$$\omega_{\alpha\beta} \equiv \tfrac{1}{2}\left[\frac{\partial u_\alpha}{\partial r_\beta} - \frac{\partial u_\beta}{\partial r_\alpha}\right]. \tag{L7}$$

$$\begin{aligned}\mathfrak{F} = \sum_{\alpha\beta\gamma\delta} \int d\vec{r} \quad & \tfrac{1}{8} \quad e_{\alpha\beta}[E_{\alpha\beta\gamma\delta} + E_{\beta\alpha\gamma\delta} + E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\delta\gamma}]e_{\gamma\delta} \\ & +\tfrac{1}{8} \quad \omega_{\alpha\beta}[E_{\alpha\beta\gamma\delta} - E_{\beta\alpha\gamma\delta} - E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\delta\gamma}]\omega_{\gamma\delta}.\end{aligned} \tag{L8}$$

$$\mathfrak{F} = \sum_{\alpha\beta\gamma\delta} \int d\vec{r}\, \tfrac{1}{2} e_{\alpha\beta}\, C_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}, \tag{L9}$$

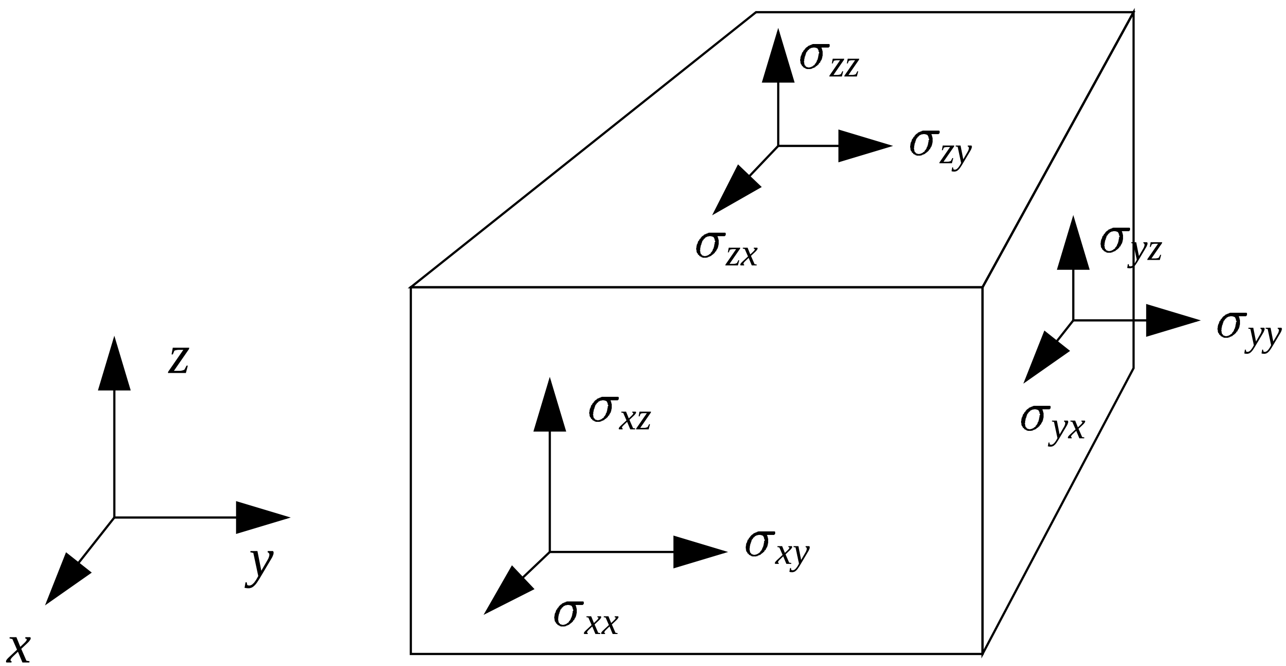
$$C_{\alpha\beta\gamma\delta} = \tfrac{1}{4}[E_{\alpha\beta\gamma\delta} + E_{\beta\alpha\gamma\delta} + E_{\alpha\beta\delta\gamma} + E_{\beta\alpha\delta\gamma}]. \tag{L10}$$

$$\alpha \leftrightarrow \beta,\ \gamma \leftrightarrow \delta \text{ and also } \alpha\beta \leftrightarrow \gamma\delta. \tag{L11}$$

$$\mathcal{F} = \sum_{\alpha\beta} \int d\vec{r}\, \tfrac{1}{2} e_{\alpha\beta}\, \sigma_{\alpha\beta}, \tag{L12}$$

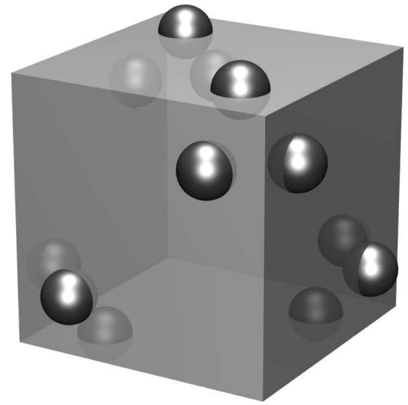
where the stress tensor is

$$\sigma_{\alpha\beta} = \sum_{\gamma\delta} C_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}. \tag{L13}$$



Equation of motion

Solids of Cubic Symmetry



C_{xyyy} vanishes because it multiplies ? ? but ? ? flips sign when $x \to -x$.

Also invariant under $x \to y \to z \to x$

Three parameters survive:

C_{xxxx}

C_{xxyy}

C_{xyxy}

Solids of Cubic Symmetry

$$\mathfrak{F} = \int d\vec{r}\,\frac{1}{2}\left\{\begin{array}{rl} C_{xxxx} & [e_{xx}^2+e_{yy}^2+e_{zz}^2] \\ +2C_{xxyy} & [e_{xx}e_{yy}+e_{yy}e_{zz}+e_{zz}e_{xx}] \\ +4C_{xyxy} & [e_{xy}^2+e_{yz}^2+e_{zx}^2] \end{array}\right\}. \tag{L14}$$

$$\begin{array}{cccccc} e_{xx} & e_{yy} & e_{zz} & 2e_{yz} & 2e_{zx} & 2e_{xy} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \tag{L15}$$

$$\begin{array}{ccccccc} C_{xxxx} & C_{xxyy} & C_{xxzz} & C_{yzxx} & C_{zxxx} & C_{xyxx} & \text{etc.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ C_{11} & C_{12} & C_{13} & C_{41} & C_{51} & C_{61} & \text{etc.} \end{array} \tag{L16}$$

Solids of Cubic Symmetry

$$\mathfrak{F} = \int d\vec{r}\, \tfrac{1}{2} \sum_{\alpha\beta=1}^{6} e_\alpha C_{\alpha\beta} e_\beta . \tag{L17}$$

Cauchy relation: $C_{44} = C_{12}$

Solids of Cubic Symmetry

Element	C_{11} (GPa)	C_{44} (GPa)	C_{12} (GPa)	Element	C_{11} (GPa)	C_{44} (GPa)	C_{12} (GPa)
Al	108	28.3	62	Li (195K)	13.4	9.6	11.3
Ar (80 K)	2.77	0.98	1.37	Mo	459	111	168
Ag	123	45.3	92	Na	7.59	4.30	6.33
Au	190	42.3	161	Ne (6 K)	1.62	0.93	0.85
Cs (78 K)	2.47	2.06	1.48	Ni	247	122	153
Ca	16	12	8	Nb	245	28.4	132
Cr	346	100	66	O (54.4 K)	2.60	0.275	2.06
Cu	169	75.3	122	Pd	224	71.6	173
C (diamond)	1040	550	170	Pt	347	76.5	251
Fe	230	117	135	Rb	2.96	1.60	2.44
Ge (undoped)	129	67.1	48	Si (undoped)	165	79.2	64
Ge (*n*-doped, 10^{19} Sb)	128.8	65.5	47.7	Si (*n*-doped, 10^{19} As)	162.2	78.7	65.4
Ge (*p*-doped, 10^{20} Ga)	118.0	65.3	39.0	Sr	14.7	5.74	9.9
He^3 (0.4 K, 24 cm^3/mole)	0.0235	0.01085	0.0197	Ta	262	82.6	156
He^4 (1.6 K, 12 cm^3/mole)	0.0311	0.0217	0.0281	Th	76	46	49
Ir	600	270	260	W	517	157	203
K	3.71	1.88	3.15	V	230	43.2	120
Kr (115 K)	2.85	1.35	1.60	Xe (156 K)	2.98	1.48	1.90
Pb	48.8	14.8	41.4				

Solids of Cubic Symmetry

$$B = \mathcal{V} \partial^2 \mathcal{F} / \partial \mathcal{V}^2$$

$$e_{xx} = e_{yy} = e_{zz} = \delta\mathcal{V}/3\mathcal{V}$$

$$\mathcal{F} = \tfrac{1}{6}\mathcal{V}\left[C_{11} + 2C_{12}\right]\left[\delta\mathcal{V}/\mathcal{V}\right]^2, \tag{L18}$$

$$B = \tfrac{1}{3}\left[C_{11} + 2C_{12}\right]. \tag{L19}$$

Isotropic Solids

Distinguish between rotating all mass points and rotating a pattern of distortion in mass points that otherwise remain fixed.

$$e_{\alpha\beta}(\vec{r}) = \sum_{\gamma\delta} R^*_{\alpha\gamma} e'_{\gamma\delta}(\vec{r}')R_{\delta\beta} \tag{L20a}$$

with

$$\vec{r}' = R\vec{r} \text{ and } R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \tag{L20b}$$

$$0 = (2C_{xyxy}+C_{xxyy}-C_{xxxx})(e_{yy}-2e_{xy}-e_{xx})(e_{yy}+2e_{xy}-e_{xx}) \tag{L21}$$

$$\Rightarrow C_{xxxx} = C_{xxyy}+2C_{xyxy}. \tag{L22}$$

$$\mathcal{F} = \frac{1}{2}\int d\vec{r}\,\lambda\Big(\sum_{\alpha} e_{\alpha\alpha}\Big)^2 + 2\mu\sum_{\alpha\beta} e^2_{\alpha\beta}. \tag{L23}$$

Kinetic energy:

Isotropic Solids

$$T = \int d\vec{r}\, \frac{1}{2} \rho |\dot{\vec{u}}(\vec{r})|^2, \tag{L24}$$

Equation of motion:

$$\rho \ddot{u}_\alpha(\vec{r}) = -\frac{\delta \mathcal{F}}{\delta u_\alpha(\vec{r})} = \sum_\beta \frac{\partial}{\partial r_\beta} \sigma_{\alpha\beta}(\vec{r}), \tag{L25}$$

$$\sigma_{\alpha\beta} = \sum_{\gamma\delta} C_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}. \tag{L26}$$

$$\int_{\mathcal{V}} d\vec{r}\, \rho \ddot{u}_\alpha = \int d\Sigma \sum n_\beta \sigma_{\beta\alpha} \tag{L27}$$

Stress figure

$$\sigma_{\alpha\beta} \quad = \quad \lambda \delta_{\alpha\beta} \sum_\gamma e_{\gamma\gamma} + 2\mu e_{\alpha\beta} \tag{L28}$$

$$\Rightarrow e_{\alpha\beta} = \frac{-\lambda\delta_{\alpha\beta}}{2\mu(3\lambda+2\mu)}\sum_{\gamma}\sigma_{\gamma\gamma} + \frac{1}{2\mu}\sigma_{\alpha\beta}. \tag{L29}$$

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) + \mu\nabla^2\vec{u}. \tag{L30}$$

$$\mathcal{S} = Ye_{zz} \tag{L31}$$

with

$$Y = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}; \tag{L32}$$

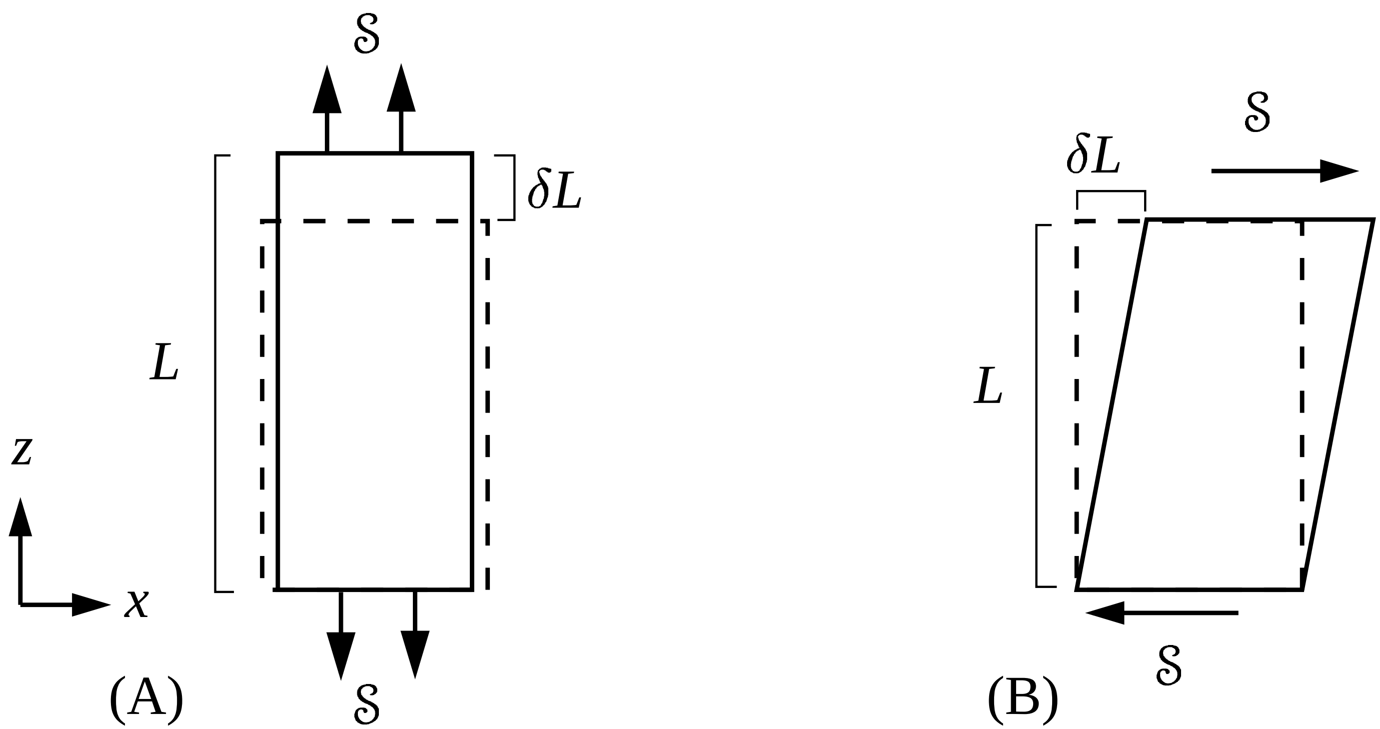
$$e_{xx} = e_{yy} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathcal{S}, \tag{L33}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{L34}$$

$$\mathcal{S} = 2Ge_{yz} = G\frac{\partial u_y}{\partial z} \tag{L35}$$

$$\mathcal{S} = \frac{\delta L}{L}Y = e_{zz}Y$$

$$\mathcal{S} = \frac{\delta L}{L}G = 2e_{xz}G$$



Isotropic Solids

Material	Young's Modulus Y (GPa)	Poisson Ratio ν
Lead (cast)	5	0.5
Tin (cast)	27	0.3
Glass	55	0.16
Aluminum (cast)	68	0.3
Copper (cast)	76	0.4
Zinc (cast)	76	0.3
Copper (soft, wrought)	100	0.4
Iron (cast)	110	0.3
Copper (hard drawn)	120	0.4
Iron (wrought)	200	0.3
Carbon steel	200	0.3
Tungsten	400	0.3

$$\Delta(\vec{r},t) = \vec{\nabla}\cdot\vec{u}(\vec{r},t) \text{ and } \vec{w}(\vec{r},t) = \vec{\nabla}\times\vec{u}(\vec{r},t). \tag{L36}$$

$$\rho\frac{\partial^2\Delta}{\partial t^2} = (\lambda+2\mu)\nabla^2\Delta, \tag{L37}$$

$$\rho\frac{\partial^2\vec{w}}{\partial t^2} = \mu\nabla^2\vec{w}, \tag{L38}$$

$\vec{u}$ is of form $\vec{u}_0 e^{i\vec{k}\cdot i\vec{r}-i\omega t}$

$$c_l = \sqrt{\frac{\lambda+2\mu}{\rho}}, \tag{L39}$$

$$c_t = \sqrt{\frac{\mu}{\rho}}. \tag{L40}$$

Director $\hat{n}$.

$$(\hat{n}\cdot\vec{\nabla})\hat{n} \quad \text{(L41a)}$$

$$\vec{\nabla}\cdot\hat{n} \quad \text{(L41b)}$$

$$\hat{n}\cdot\vec{\nabla}\times\hat{n}. \quad \text{(L41c)}$$

$$\frac{\partial n_\alpha}{\partial r_\beta}\frac{\partial n_\gamma}{\partial r_\delta}, \quad \text{(L42)}$$

$$\mathfrak{F} = \int d\vec{r}\,\mathfrak{F}(\vec{r}) = \tfrac{1}{2}\int d\vec{r}\sum_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}\frac{\partial n_\alpha}{\partial r_\beta}\frac{\partial n_\gamma}{\partial r_\delta}. \quad \text{(L43)}$$

$$0 = \frac{\partial}{\partial r_\alpha}1 = \frac{\partial}{\partial r_\alpha}(\hat{n}\cdot\hat{n}) \quad \text{(L44)}$$

$$= 2n_z\frac{\partial}{\partial r_\alpha}n_z = 2\frac{\partial}{\partial r_\alpha}n_z \quad \text{(L45)}$$

$$\frac{\partial n_\gamma}{\partial r_\delta} \rightarrow \frac{\partial n_\gamma}{\partial r_\delta} + \theta[\sum_\beta \frac{\partial n_\gamma}{\partial r_\beta} R_{\beta\delta} - R_{\gamma\beta}\frac{\partial n_\beta}{\partial r_\delta}] \tag{L46}$$

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{L47}$$

$$\begin{aligned} 0 = & \sum_{\alpha\gamma\delta}\left[\frac{\partial n_\alpha}{\partial y}\frac{\partial n_\gamma}{\partial r_\delta}C_{\alpha x\gamma\delta} - \frac{\partial n_\alpha}{\partial x}\frac{\partial n_\gamma}{\partial r_\delta}C_{\alpha y\gamma\delta}\right] \\ - & \sum_{\beta\gamma\delta}\left[\frac{\partial n_x}{\partial r_\beta}\frac{\partial n_\gamma}{\partial r_\delta}C_{y\beta\gamma\delta} - \frac{\partial n_y}{\partial r_\beta}\frac{\partial n_\gamma}{\partial r_\delta}C_{x\beta\gamma\delta}\right]. \end{aligned} \tag{L48}$$

$$\frac{\partial n_x}{\partial z}\frac{\partial n_y}{\partial y}, \tag{L49}$$

$$0 = -C_{yzyy} + C_{yxxz} + C_{xyxz}. \tag{L50}$$

$$\left[\frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y}\right]^2 \tag{L51a}$$

$$\left[\frac{\partial n_x}{\partial z}\right]^2 + \left[\frac{\partial n_y}{\partial z}\right]^2 \tag{L51b}$$

$$\left[\frac{\partial n_y}{\partial x} - \frac{\partial n_x}{\partial y}\right]^2 \tag{L51c}$$

$$\left[\frac{\partial n_y}{\partial x} - \frac{\partial n_x}{\partial y}\right]\left[\frac{\partial n_y}{\partial y} + \frac{\partial n_x}{\partial x}\right] \tag{L51d}$$

$$\frac{\partial n_y}{\partial x}\frac{\partial n_x}{\partial y} - \frac{\partial n_y}{\partial y}\frac{\partial n_x}{\partial x}. \tag{L51e}$$

$$(\vec{\nabla} \cdot \hat{n})^2 \tag{L52a}$$

$$|\hat{n} \times (\vec{\nabla} \times \hat{n})|^2 \tag{L52b}$$

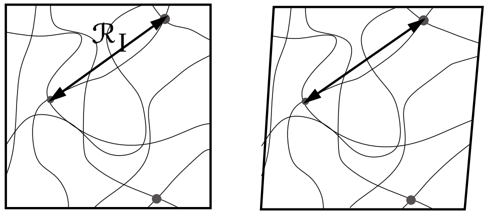
$$(\hat{n} \cdot (\vec{\nabla} \times \hat{n}))^2 \tag{L52c}$$

$$\hat{n} \cdot (\vec{\nabla} \times \hat{n})\vec{\nabla} \cdot \hat{n} \tag{L52d}$$

$$\frac{1}{2} \qquad \vec{\nabla}\cdot\left[(\hat{n}\cdot\vec{\nabla})\hat{n}-\hat{n}(\vec{\nabla}\cdot\hat{n})\right]. \tag{L52e}$$

$$\mathfrak{F}=\underbrace{\frac{K_1}{2}(\vec{\nabla}\cdot\hat{n})^2}_{\text{splay}} \quad +\underbrace{\frac{K_2}{2}(\hat{n}\cdot(\vec{\nabla}\times\hat{n}))^2}_{\text{twist}} \quad +\underbrace{\frac{K_3}{2}(\hat{n}\times(\vec{\nabla}\times\hat{n}))^2}_{\text{bend}}. \tag{L53}$$

$$\mathcal{F} = \mathcal{F}_0 + k_B T \left[\sum_{j=1}^{N_p} \frac{\mathcal{R}_j^2}{\mathcal{R}_I{}^2} + N \frac{\mathcal{R}_I^2}{(\mathcal{V})^{2/3}} - \mathcal{V}|B|n^2 + \mathcal{V}Cn^3 + \ldots \right]. \tag{L54}$$



$$\mathcal{R}_j^\alpha \to \mathcal{R}_j^\alpha + \sum_\beta \mathcal{R}_j^\beta \frac{\partial u_\alpha}{\partial r_\beta}. \tag{L55}$$

$$\mathcal{V} = \mathcal{V} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \left(\delta_{x\alpha} + \frac{\partial u_\alpha}{\partial x} \right) \left(\delta_{y\beta} + \frac{\partial u_\beta}{\partial y} \right) \left(\delta_{z\gamma} + \frac{\partial u_\gamma}{\partial z} \right) \tag{L56}$$

$$\Rightarrow \sum_\alpha \frac{\partial u_\alpha}{\partial r_\alpha} = \frac{1}{2} \sum_{\alpha\beta} \left[\frac{\partial u_\alpha}{\partial r_\beta} \frac{\partial u_\beta}{\partial r_\alpha} - \frac{\partial u_\alpha}{\partial r_\alpha} \frac{\partial u_\beta}{\partial r_\beta} \right]. \tag{L57}$$

$$\mathcal{F} = \frac{k_B T}{\mathcal{R}_{\rm I}^2} \sum_j \sum_\alpha \left[(\mathcal{R}_j^\alpha)^2 + 2\sum_\beta \mathcal{R}_j^\alpha \frac{\partial u_\alpha}{\partial r_\beta} \mathcal{R}_j^\beta + \sum_{\beta\beta'} \frac{\partial u_\alpha}{\partial r_\beta} \frac{\partial u_\alpha}{\partial r_{\beta'}} \mathcal{R}_j^\beta \mathcal{R}_j^{\beta'} \right]. \qquad \text{(L58)}$$

$$\sum_{j=1}^{N_{\rm p}} \mathcal{R}_j^\alpha \mathcal{R}_j^\beta = N_{\rm p} \frac{\mathcal{R}_{\rm I}^2}{3} \delta_{\alpha\beta} \qquad \text{(L59)}$$

$$\Rightarrow \mathcal{F} = \frac{k_B T N_{\rm p}}{3} [3 + 2\sum_\beta \frac{\partial u_\beta}{\partial r_\beta} + \sum_{\alpha\beta} \frac{\partial u_\alpha}{\partial r_\beta} \frac{\partial u_\alpha}{\partial r_\beta}] \qquad \text{(L60)}$$

$$\Rightarrow \mathcal{F} = \frac{k_B T N_{\rm p}}{3} \left[\sum_{\alpha\beta} \frac{\partial u_\alpha}{\partial r_\beta} \frac{\partial u_\alpha}{\partial r_\beta} + \sum_{\alpha\beta} \left(\frac{\partial u_\alpha}{\partial r_\beta} \frac{\partial u_\beta}{\partial r_\alpha} - \frac{\partial u_\alpha}{\partial r_\alpha} \frac{\partial u_\beta}{\partial r_\beta} \right) \right] \qquad \text{(L61)}$$

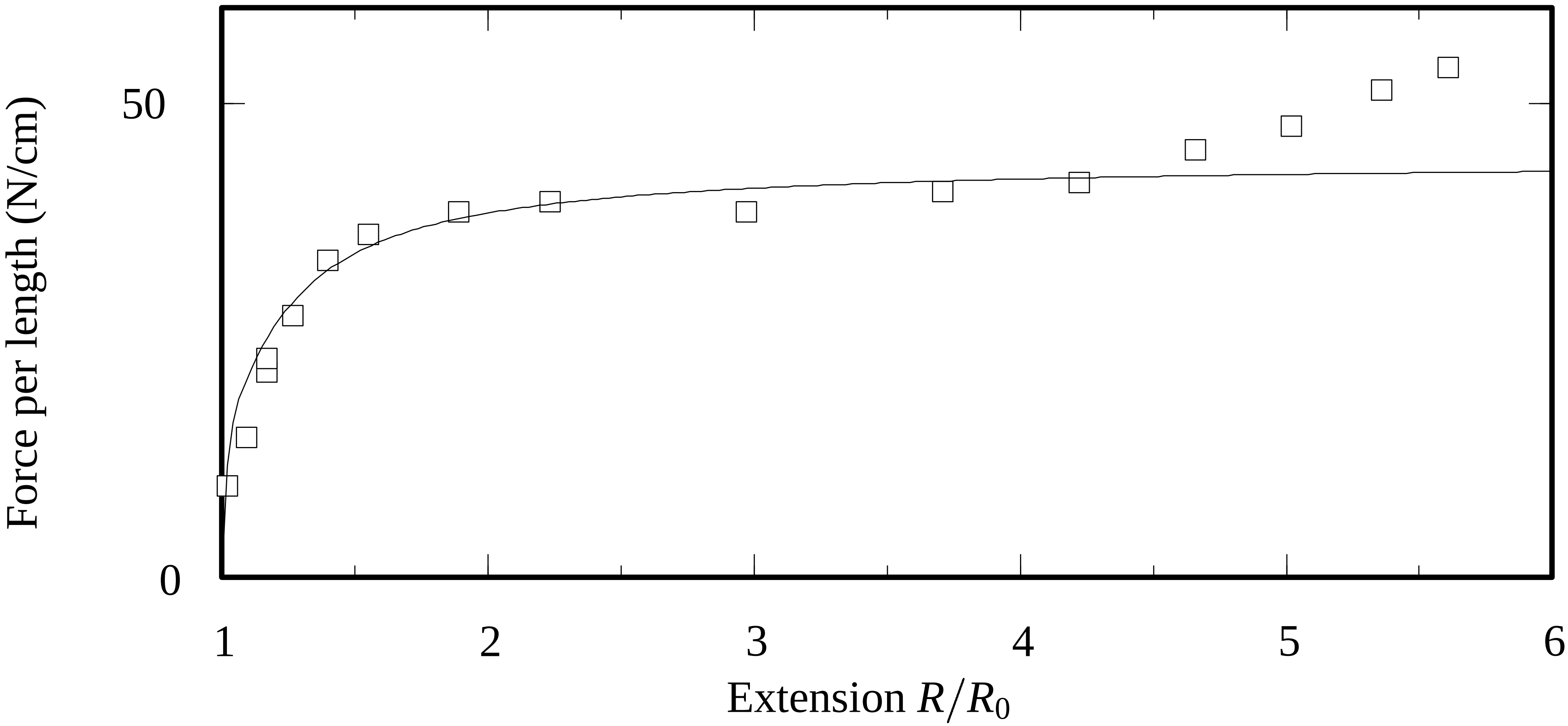
$$= \frac{k_B T N_{\rm p}}{3} \sum_{\alpha\beta} \left[2e_{\alpha\beta}^2 - (\sum_\alpha e_{\alpha\alpha})^2 \right] \qquad \text{(L62)}$$

$$= \frac{2k_B T N_{\rm p}}{3} \sum_{\alpha\beta} e_{\alpha\beta}^2. \qquad \text{(L63)}$$

Rubber

$$\mathfrak{F} = \frac{2k_BTN_\mathrm{p}}{3}\left[\Big(\sum_{\alpha\beta}[e_{\alpha\beta}+\delta_{\alpha\beta}]^2\Big)-3\right]. \qquad \text{(L64)}$$

$$\mathfrak{F} = \frac{2k_BTN_\mathrm{p}}{3}\left[2\Big(\frac{R}{R_0}\Big)^2+\left\{\Big(\frac{R_0}{R}\Big)^2\right\}^2-3\right]. \qquad \text{(L65)}$$



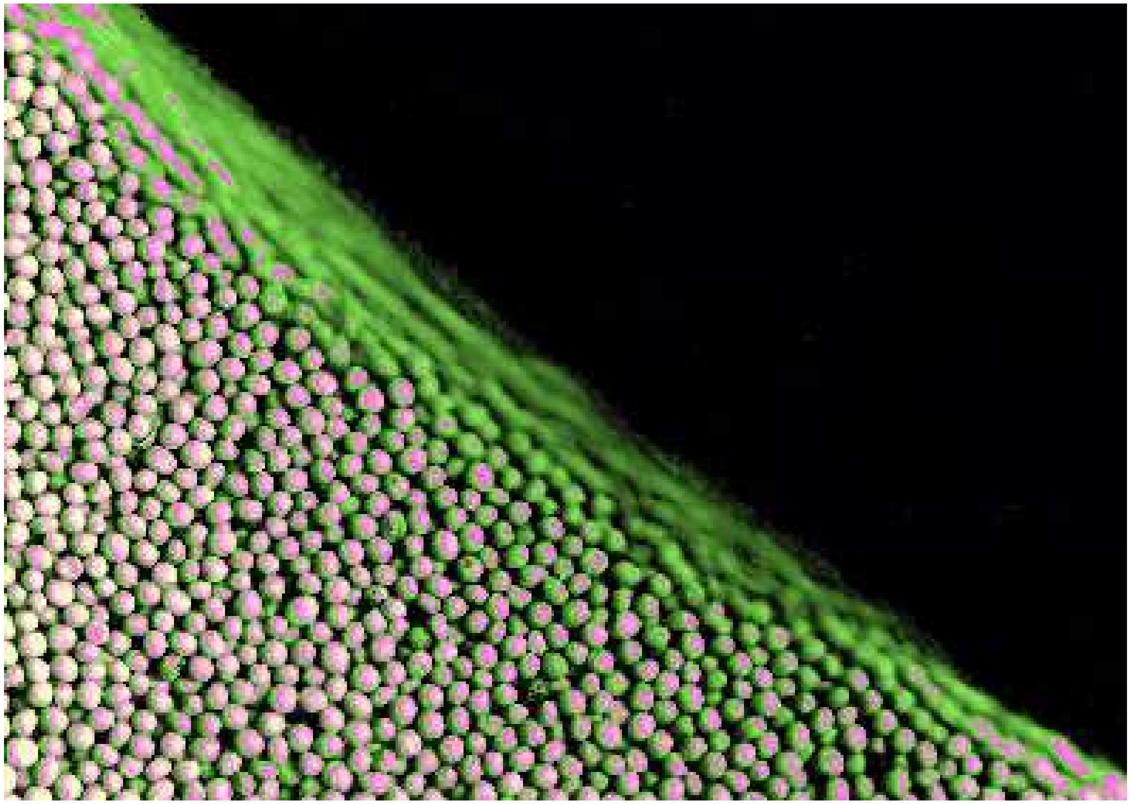


Figure 1: Avalanche in mustard seeds: Jaeger, University of Chicago

University of Chicago Granular Group

Duke University Granular Page