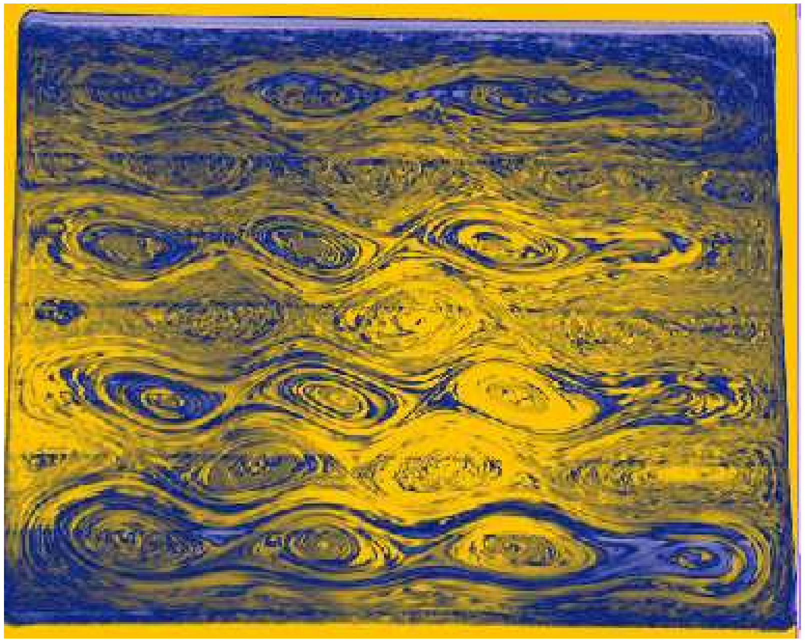
# Fluid Mechanics





# Euler's Equation



$$\vec{v}(\vec{r}+\vec{v}dt,t+dt) = \vec{v}(\vec{r},t)+\vec{f}(\vec{r},t)dt/\rho \tag{L1}$$

$$\Rightarrow \quad \frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{\vec{f}}{\rho}. \tag{L2}$$

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}+\frac{\vec{\nabla}P}{\rho} = 0. \tag{L3}$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\rho\vec{v}. \tag{L4}$$

$$0 = \frac{\partial \rho v_\alpha}{\partial t} - v_\alpha\frac{\partial \rho}{\partial t}+\rho\sum_\beta v_\beta\frac{\partial}{\partial r_\beta}v_\alpha+\frac{\partial}{\partial r_\alpha}P \tag{L5}$$

$$= \frac{\partial \rho v_\alpha}{\partial t} + v_\alpha\sum_\beta\frac{\partial}{\partial r_\beta}\rho v_\beta+\rho\sum_\beta v_\beta\frac{\partial}{\partial r_\beta}v_\alpha+\frac{\partial}{\partial r_\alpha}P \tag{L6}$$

$$= \frac{\partial \rho v_\alpha}{\partial t} + \sum_\beta\frac{\partial}{\partial r_\beta}\{\rho v_\alpha v_\beta+\delta_{\alpha\beta}P\}. \tag{L7}$$

# Euler's Equation



$$\sigma_{\alpha\beta} = -\rho v_\alpha v_\beta - \delta_{\alpha\beta} P, \tag{L8}$$

$$\frac{\partial \rho v_\alpha}{\partial t} = \sum_\beta \frac{\partial}{\partial r_\beta} \sigma_{\alpha\beta}. \tag{L9}$$

$$\vec{\nabla} \cdot \vec{v} = 0, \tag{L10}$$

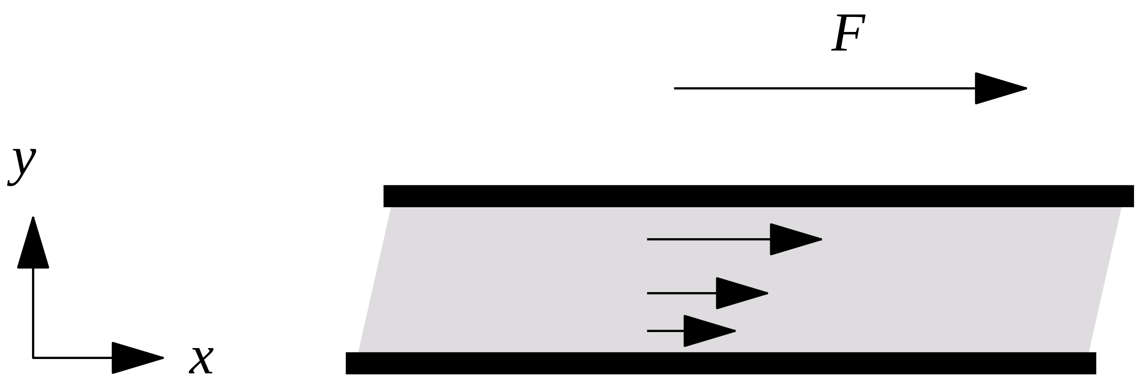



Figure 1: When liquid is sheared between two plates, the force is proportional to the shearing speed and is inversely proportional to the separation $d$.

$$\frac{F}{A} = \eta \frac{\partial v_x}{\partial y}, \qquad \text{(L11)}$$

# Navier–Stokes Equation



| Gas | $\eta$ (g/[cm·sec]) | Liquid | $\eta$ (g/[cm·sec]) |
|---|---|---|---|
| He | $1.99 \cdot 10^{-4}$ | $NH_3$ | $14 \cdot 10^{-4}$ |
| Ne | $3.17 \cdot 10^{-4}$ | $H_2O$ | $82 \cdot 10^{-4}$ |
| Ar | $2.27 \cdot 10^{-4}$ | $CO_2$ | $6.0 \cdot 10^{-4}$ |
| Kr | $2.55 \cdot 10^{-4}$ | Hg | $160 \cdot 10^{-4}$ |
| Xe | $2.33 \cdot 10^{-4}$ | Glycerine | $85\,000 \cdot 10^{-4}$ |
| $H_2$ | $0.89 \cdot 10^{-4}$ | | |
| $N_2$ | $1.79 \cdot 10^{-4}$ | | |
| $O_2$ | $2.07 \cdot 10^{-4}$ | | |
| $F_2$ | $2.36 \cdot 10^{-4}$ | | |
| $Cl_2$ | $1.37 \cdot 10^{-4}$ | | |
| CO | $1.78 \cdot 10^{-4}$ | | |
| $CO_2$ | $1.50 \cdot 10^{-4}$ | | |
| Air | $1.85 \cdot 10^{-4}$ | | |

# Navier–Stokes Equation

$$\sigma'_{\alpha\beta} = \eta\left[\frac{\partial v_\alpha}{\partial r_\beta} + \frac{\partial v_\beta}{\partial r_\alpha}\right] + \left[\zeta - \frac{2}{3}\eta\right]\delta_{\alpha\beta}\sum_\gamma \frac{\partial v_\gamma}{\partial r_\gamma}; \tag{L12}$$

$$\sigma_{\alpha\beta} = -\rho v_\alpha v_\beta - \delta_{\alpha\beta}P + \eta\left[\frac{\partial v_\alpha}{\partial r_\beta} + \frac{\partial v_\beta}{\partial r_\alpha}\right]. \tag{L13}$$

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}P + \eta\nabla^2\vec{v}. \tag{L14}$$

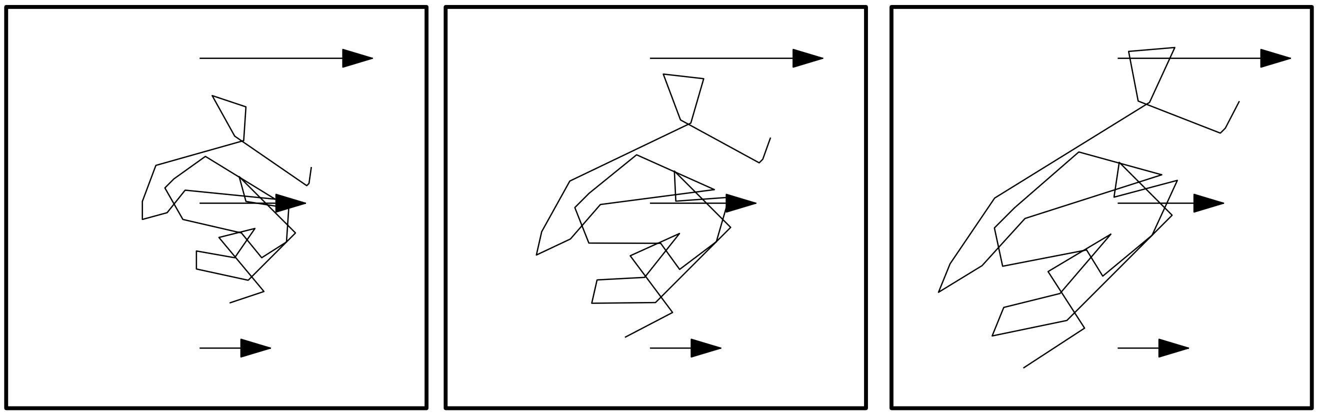


Figure 2:

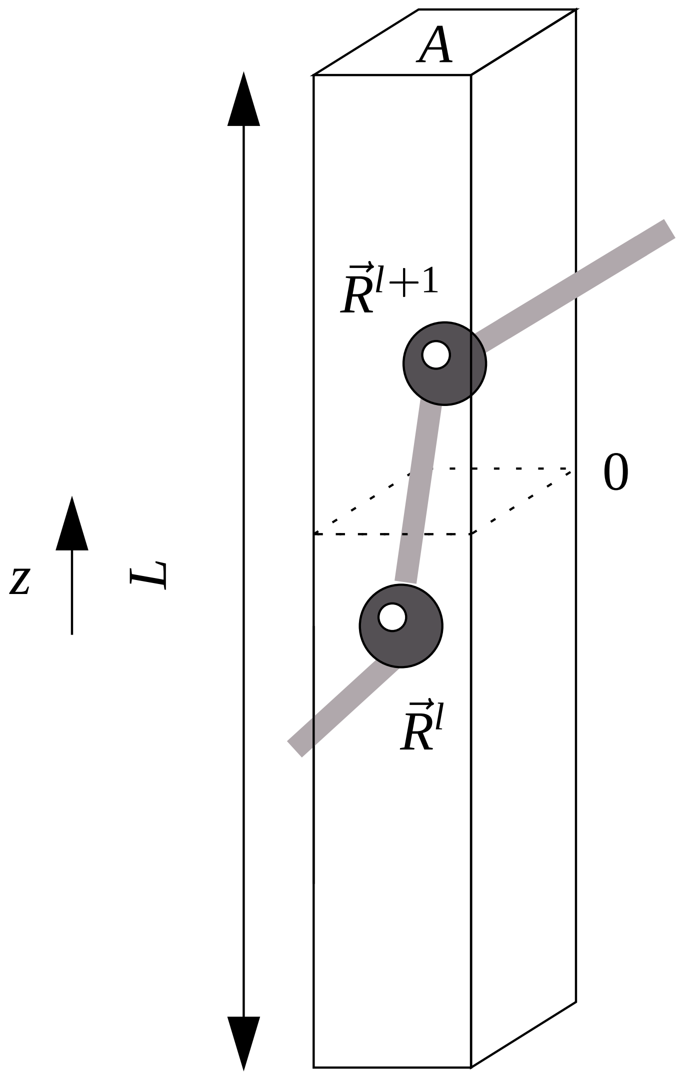



Figure 3:

$$g(\vec{R}^l,\vec{R}^{l+1}) = \frac{1}{\mathcal{V}} g(\vec{R}^{l+1}-\vec{R}^l). \tag{L15}$$

$$\sigma_{z\beta} = \frac{1}{A}\int d\vec{R}^l\, d\vec{R}^{l+1}\, \frac{1}{\mathcal{V}} g(\vec{R}^{l+1}-\vec{R}^l)\,\theta(R_z^{l+1})\theta(-R_z^l)F_\beta^{l+1,l} \tag{L16}$$

$$= \frac{1}{A\mathcal{V}}\int d\vec{s}\,d\vec{t}\,g(\vec{s})\,\theta(s_z/2+t_z)\theta(s_z/2-t_z)F_\beta^{l+1,l} \tag{L17}$$

$$= \frac{1}{\mathcal{V}}\int d\vec{s}\,g(\vec{s})\,s_z\theta(s_z)F_\beta^{l+1,l} \tag{L18}$$

$$= \frac{1}{\mathcal{V}}\left\langle [R_z^{l+1}-R_z^l]\theta(R_z^{l+1}-R_z^l)F_\beta^{l+1,l}\right\rangle \tag{L19}$$

$$\frac{1}{\mathcal{V}}\left\langle [R_z^{l+1}-R_z^l]\theta(R_z^l-R_z^{l+1})F_\beta^{l+1,l}\right\rangle. \tag{L20}$$

$$\sigma_{z\beta}^{l,l+1} = \frac{1}{\mathcal{V}}\left\langle [R_z^{l+1}-R_z^l]F_\beta^{l+1,l}\right\rangle \tag{L21}$$

$$= \quad \frac{1}{\mathcal{V}}\left\langle [R_z^{l+1}F_\beta^{l+1,l}\right\rangle + \frac{1}{\mathcal{V}}\left\langle R_z^l F_\beta^{l,l+1}\right\rangle \tag{L22}$$

$$= \quad \frac{1}{\mathcal{V}}\left\langle [R_z^{l+1}F_\beta^{l+1,l}\right\rangle + \frac{1}{\mathcal{V}}\left\langle R_z^{l-1} F_\beta^{l-1,l}\right\rangle . \tag{L23}$$

$$\sigma_{\alpha\beta} = \frac{1}{\mathcal{V}}\sum_{ll'}\left\langle R_\alpha^{l'} F_\beta^{l',l}\right\rangle . \tag{L24}$$

$$\ddot{\vec{R}}^l = \frac{1}{m}\sum_{l'}\vec{F}^{l',l} - b(\dot{\vec{R}}^l - \vec{v}) + \vec{\xi}^l . \tag{L25}$$

$$\langle \xi_\alpha(0)\xi_\beta(t)\rangle = \frac{2b\delta_{\alpha\beta}k_B T\delta(t)}{m}. \tag{L26}$$

$$\dot{\vec{R}}^l = \vec{v} + \frac{\mathcal{K}}{bm}[\vec{R}^{l+1} - 2\vec{R}^l + \vec{R}^{l-1}] + \frac{\vec{\xi}^l}{b}. \tag{L27}$$

$$v_\alpha = \vec{v}_\alpha^0 + \sum_\beta W_{\alpha\beta}R_\beta^l . \tag{L28}$$

$$\dot{\vec{R}}^l = \vec{v}^0 + W\vec{R}^l + \frac{\mathcal{K}}{bm}[\vec{R}^{l+1} - 2\vec{R}^l + \vec{R}^{l-1}] + \frac{\vec{\xi}^l}{b}. \tag{L29}$$

$$\vec{\psi}^k = \frac{1}{\sqrt{N}}\sum_{l=1}^{N} e^{2\pi i l k/N}[\vec{R}^l - \vec{v}^0 t]. \tag{L30}$$

$$\dot{\vec{\psi}}^k = \{W - \omega_k\}\,\vec{\psi}^k + \frac{\vec{\xi}^k}{b} \tag{L31}$$

with

$$\omega_k = \frac{2\mathcal{K}}{mb}(1 - \cos[2\pi k/N]). \tag{L32}$$

If $W$ is independent of time, one can write

$$\vec{\psi}^k = \int_{-\infty}^{t} dt'\, e^{-(t'-t)[W-\omega_k]}\,\frac{\xi^k(t')}{b}. \tag{L33}$$

$$\psi_\alpha^{(0)k} = \int_{-\infty}^{t} dt'\, e^{(t'-t)\omega_k}\,\frac{\xi^k(t')}{b} \tag{L34}$$

# Polymeric Solutions

$$\Rightarrow \quad \left\langle \psi_\alpha^{(0)k}(t)\psi^{(0)k^*}{}_\beta(t')\right\rangle = e^{-|t-t'|\omega_k}\frac{k_BT}{mb\omega_k}\delta_{\alpha\beta}. \qquad \text{(L35)}$$

$$\psi_\alpha^k \quad \approx \quad \psi_\alpha^{(0)k} + \int_{-\infty}^t dt' \sum_\beta W_\beta(t')\psi_\beta^{(0)k}(t') \qquad \text{(L36)}$$

$$\Rightarrow \quad \langle \psi_\alpha^k(t)\psi_\beta^{*k}(t)\rangle \approx \frac{k_BT}{mb\omega_k}\delta_{\alpha\beta}$$

$$+ \int_{-\infty}^t dt' \sum_{\alpha'} \left\langle \psi_\alpha^{(0)k}(t)W_{\beta\alpha'}(t')\psi_{\alpha'}^{(0)k*}(t')\right\rangle$$

$$+ \int_{-\infty}^t dt' \sum_{\alpha'} \left\langle \psi_\beta^{(0)k*}(t)W_{\alpha\alpha'}(t')\psi_{\alpha'}^{(0)k}(t')\right\rangle. \qquad \text{(L37)}$$

$$= \quad \frac{k_BT}{mb\omega_k}\left\{\delta_{\alpha\beta} + \int_{-\infty}^t dt'\, e^{-(t-t')\omega_k}[W_{\beta\alpha}(t') + W_{\alpha\beta}(t')]\right\}. \qquad \text{(L38)}$$

$$\sigma_{\alpha\beta} \quad = \quad \frac{1}{\mathcal{V}}\sum_{ll'}\left\langle F_\beta^{l,l'}R_\alpha^l\right\rangle = -\frac{\mathcal{K}}{\mathcal{V}}\sum_l \left\langle R_\alpha^l(R_\beta^{l+1} - 2R_\beta^l + R_\beta^{l-1})\right\rangle \qquad \text{(L39)}$$

$$= \frac{\mathcal{K}}{\mathcal{V}} \sum_{k=1}^{N-1} (2 - 2\cos 2\pi k/N) \left\langle \psi_\alpha^k \psi_\beta^{k*} \right\rangle \tag{L40}$$

$$= \frac{mb}{\mathcal{V}} \sum_{k=1}^{N-1} \omega_k \left\langle \psi_\alpha^k \psi_\beta^{k*} \right\rangle \tag{L41}$$

$$= \frac{k_B T}{\mathcal{V}} \sum_{k=1}^{N-1} \left[ \delta_{\alpha\beta} + \frac{W_{\alpha\beta} + W_{\beta\alpha}}{\omega_k} \right] \tag{L42}$$

$$= \frac{k_B T}{\mathcal{V}} \left[ N\delta_{\alpha\beta} + 2 \sum_{k=1}^{N/2} \frac{W_{\alpha\beta} + W_{\beta\alpha}}{\omega_k} \right] \tag{L43}$$

$$\approx \frac{k_B T}{\mathcal{V}} \left[ N\delta_{\alpha\beta} + 2 \sum_{k=1}^{\infty} \frac{W_{\alpha\beta} + W_{\beta\alpha}}{\frac{2\mathcal{K}}{mb} \frac{1}{2} \left( \frac{2\pi k}{N} \right)^2} \right] \tag{L44}$$

$$= \frac{k_B T}{\mathcal{V}} \left[ N\delta_{\alpha\beta} + (W_{\alpha\beta} + W_{\beta\alpha}) \frac{mbN^2}{12\mathcal{K}} \right] \tag{L45}$$

$$\sigma_{xy} = \frac{k_B T}{\mathcal{V}} mb \frac{N^2}{12\mathcal{K}} \frac{\partial v_x}{\partial y}. \tag{L46}$$

$$\delta\eta = \frac{c}{N} k_B T mb \frac{N^2}{12\mathcal{K}} \tag{L47}$$

$$= \frac{c}{N} mb \frac{N^2 a^2}{12} \tag{L48}$$

# Polymeric Solutions



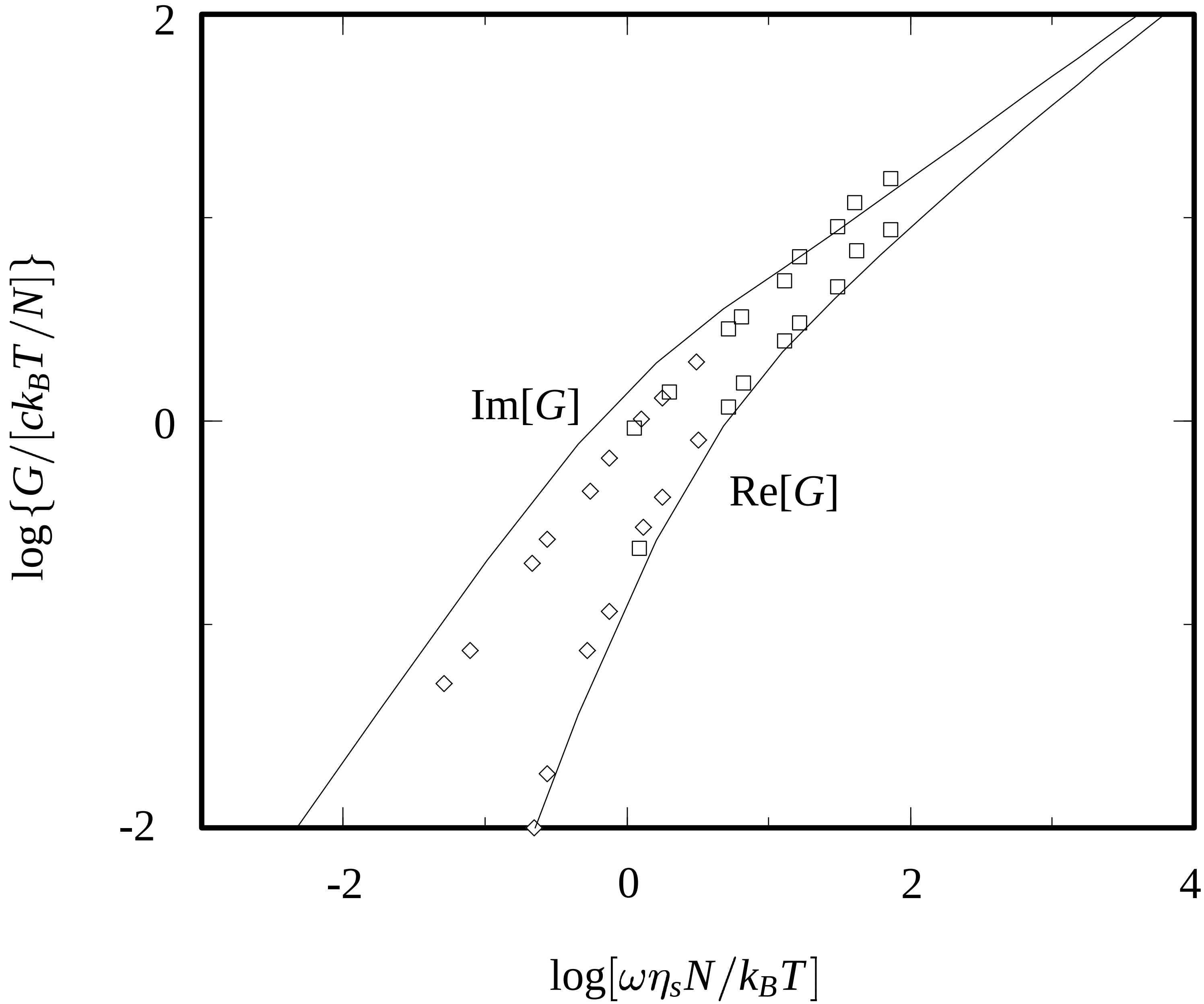



Figure 4: [Source: Ferry (1980), p. 197.]

$$\sigma_{xy} = \sum_{k=1}^{N-1} \frac{W_0 k_B T}{\mathcal{V}(\omega_k^2 + \omega^2)} [\omega_k \cos \omega t + \omega \sin \omega t]. \tag{L49}$$

$$G(\omega) = \frac{k_B T}{\mathcal{V}} \sum_{k=1}^{N-1} \frac{\omega(\omega + i\omega_k)}{\omega_k^2 + \omega^2}. \tag{L50}$$

$$\det|\sigma - \lambda I| = -\lambda^3 + \lambda^2 I_1 + \lambda I_2 + I_3 = 0 \quad \text{(L51a)}$$

with

$$I_1 = \sum_{\alpha} \sigma_{\alpha\alpha} \quad \text{(L51b)}$$

$$I_2 = \frac{1}{2} \sum_{\alpha\beta} \{\sigma_{\alpha\beta}\sigma_{\alpha\beta} - \sigma_{\alpha\alpha}\sigma_{\beta\beta}\} \quad \text{(L51c)}$$

$$I_3 = \det|\sigma|. \quad \text{(L51d)}$$

$$\sigma = \frac{1}{3} \sum_{\alpha} \sigma_{\alpha\alpha} \quad \text{(L52)}$$

$$s_{\alpha\beta} = \sigma_{\alpha\beta} - \sigma\delta_{\alpha\beta}. \quad \text{(L53)}$$

$$J_2 = \frac{1}{2} \sum_{\alpha\beta} s_{\alpha\beta} s_{\alpha\beta} \quad \text{(L54a)}$$

$$J_3 = \det|s|. \tag{L54b}$$

$$\sqrt{J_2} = \kappa. \tag{L55}$$

$$\dot{e}^{\mathrm{p}}_{\alpha\beta} = \begin{cases} w[\sqrt{J_2} - \kappa]s_{\alpha\beta} & \text{if } \sqrt{J_2} - \kappa > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{L56}$$

$$W = \int dt' \sum_{\alpha\beta} \dot{e}^{\mathrm{p}}_{\alpha\beta} \sigma_{\alpha\beta}. \tag{L57}$$

$$de^{\mathrm{p}}_{\alpha\beta} = C ds_{\alpha\beta}. \tag{L58}$$

$$dW = C \sum_{\alpha\beta} \sigma_{\alpha\beta} ds_{\alpha\beta} \tag{L59}$$

$$= C \sum_{\alpha\beta} s_{\alpha\beta} ds_{\alpha\beta} \tag{L60}$$

$$= \quad C dJ_2, \tag{L61}$$

$$d\kappa = \kappa' C\, dJ_2. \tag{L62}$$

$$C \quad = \quad \frac{1}{2\kappa'\sqrt{J_2}} \tag{L63}$$

$$\Rightarrow de^{\mathrm{p}}_{\alpha\beta} \quad = \quad \frac{ds_{\alpha\beta}}{2\kappa'\sqrt{J_2}}. \tag{L64}$$

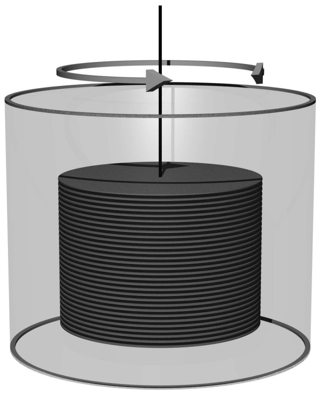


Figure 5:

$$\omega = \sqrt{\frac{\mathcal{K}}{I_0 + I_F}}. \tag{L65}$$

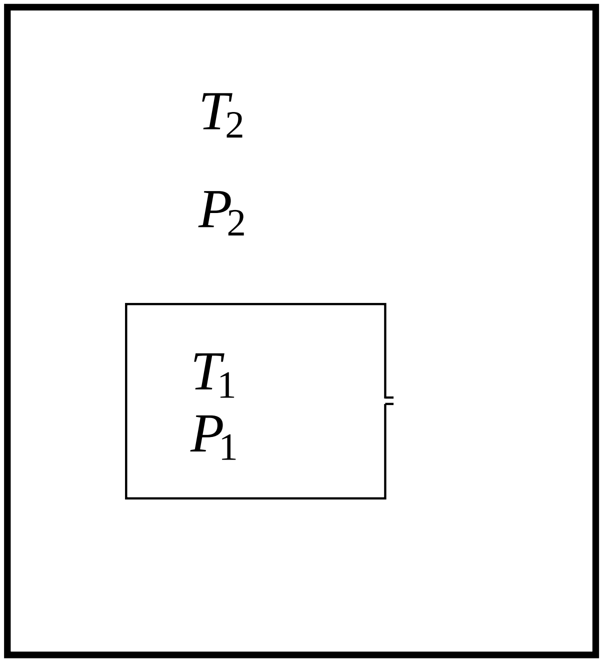



Figure 6:

$$0 = \frac{\partial G}{\partial N_1} = \frac{\partial G_1(N_1) + G_2(N - N_1)}{\partial N_1}\Big|_{TP} \tag{L66}$$

$$\Rightarrow \frac{\partial G_1}{\partial N_1} = \frac{\partial G_2}{\partial N_2} \Rightarrow \mu_1(T_1, P_1) = \mu_2(T_2, P_2). \tag{L67}$$

$$\frac{\partial \mathcal{E}_1(S_1, \mathcal{V}_1)}{\partial S_1} = \frac{\partial \mathcal{E}_2(S_2, \mathcal{V}_2)}{\partial S_2} \Rightarrow T_1 = T_2; \tag{L68}$$

$$\frac{\partial \mu_2}{\partial T_2} \Delta T + \frac{\partial \mu_2}{\partial P_2} \Delta P = 0 \tag{L69}$$

$$\Rightarrow \quad s\Delta T = \frac{1}{\rho} \Delta P \Rightarrow \frac{\Delta P}{\Delta T} = \rho s. \tag{L70}$$

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{\vec{\nabla}\mu}{m}. \tag{L71}$$

$$d\mu = \frac{\mathcal{V}}{N}dP - \frac{S}{N}dT, \tag{L72}$$

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{\vec{\nabla}P}{\rho} + s\vec{\nabla}T. \tag{L73}$$

$$\rho_s \left\{ \frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s \right\} + \rho_n \left\{ \frac{\partial \vec{v}_n}{\partial t} + (\vec{v}_n \cdot \vec{\nabla})\vec{v}_n \right\} = -\vec{\nabla}P + \eta\nabla^2\vec{v}_n. \tag{L74}$$

# Second Sound



$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho_n \vec{v}_n + \rho_s \vec{v}_s) = 0 \tag{L75}$$

$$\frac{\partial \rho s}{\partial t} = -\vec{\nabla} \cdot \rho s \vec{v}_n \tag{L76}$$

$$\frac{\partial \vec{v}_s}{\partial t} = -\frac{\vec{\nabla} P}{\rho} + s \vec{\nabla} T \tag{L77}$$

$$\rho_s \frac{\partial \vec{v}_s}{\partial t} + \rho_n \frac{\partial \vec{v}_n}{\partial t} = -\vec{\nabla} P. \tag{L78}$$

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 P. \tag{L79}$$

$$\frac{\partial s}{\partial t} = \frac{1}{\rho} \frac{\partial s \rho}{\partial t} - \frac{s}{\rho} \frac{\partial \rho}{\partial t} \tag{L80}$$

$$= \frac{-1}{\rho} \vec{\nabla} \cdot \rho s \vec{v}_n + \frac{s}{\rho} \vec{\nabla} \cdot (\rho_n \vec{v}_n + \rho_s \vec{v}_s) \tag{L81}$$

$$= \quad \frac{s\rho_s}{\rho}\vec{\nabla}\cdot(\vec{v}_s-\vec{v}_n). \qquad \text{(L82)}$$

Solving Eqs. (L77) and (78) for $\partial(\vec{v}_s-\vec{v}_n)/\partial t$ gives

$$\frac{\partial}{\partial t}(\vec{v}_s-\vec{v}_n) = s\frac{\rho}{\rho_n}\vec{\nabla}T \qquad \text{(L83)}$$

$$\Rightarrow \frac{\partial^2 s}{\partial t^2} = s^2\frac{\rho_s}{\rho_n}\nabla^2 T. \qquad \text{(L84)}$$

$$\frac{\partial\rho}{\partial P}\Big|_T\frac{\partial^2 P^{(1)}}{\partial t^2}+\frac{\partial\rho}{\partial T}\Big|_P\frac{\partial^2 T^{(1)}}{\partial t^2} = \nabla^2 P^{(1)} \qquad \text{(L85)}$$

$$\frac{\partial s}{\partial P}\Big|_T\frac{\partial^2 P^{(1)}}{\partial t^2}+\frac{\partial s}{\partial T}\Big|_P\frac{\partial^2 T^{(1)}}{\partial t^2} = s^2\frac{\rho_s}{\rho_n}\nabla^2 T^{(1)}. \qquad \text{(L86)}$$

$$\frac{\partial\rho}{\partial P}\Big|_T P^{(1)}+\frac{\partial\rho}{\partial T}\Big|_P T^{(1)} = c^{-2}P^{(1)} \qquad \text{(L87a)}$$

$$\frac{\partial s}{\partial P}\Big|_T P^{(1)}+\frac{\partial s}{\partial T}\Big|_P T^{(1)} = c^{-2}s^2\frac{\rho_s}{\rho_n}T^{(1)}. \qquad \text{(L87b)}$$

$$\left(1-\frac{c^{-2}s^2\rho_s/\rho_n}{\frac{\partial s}{\partial T}\,|_P}\right)\left(1-\frac{c^{-2}}{\frac{\partial \rho}{\partial P}\,|_T}\right)=\frac{\frac{\partial s}{\partial P}\,|_T\,\frac{\partial \rho}{\partial T}\,|_P}{\frac{\partial \rho}{\partial P}\,|_T\,\frac{\partial s}{\partial T}\,|_P} \tag{L88}$$

$$= \frac{C_P-C_V}{C_P} \tag{L89}$$

$$\approx 0. \tag{L90}$$

$$c_1 = \sqrt{\frac{\partial P}{\partial \rho}\,|_T} \tag{L91}$$

and

$$c_2 = \sqrt{\frac{Ts^2}{C_P}\frac{\rho_s}{\rho_n}} \tag{L92}$$

# Origin of Superfluidity

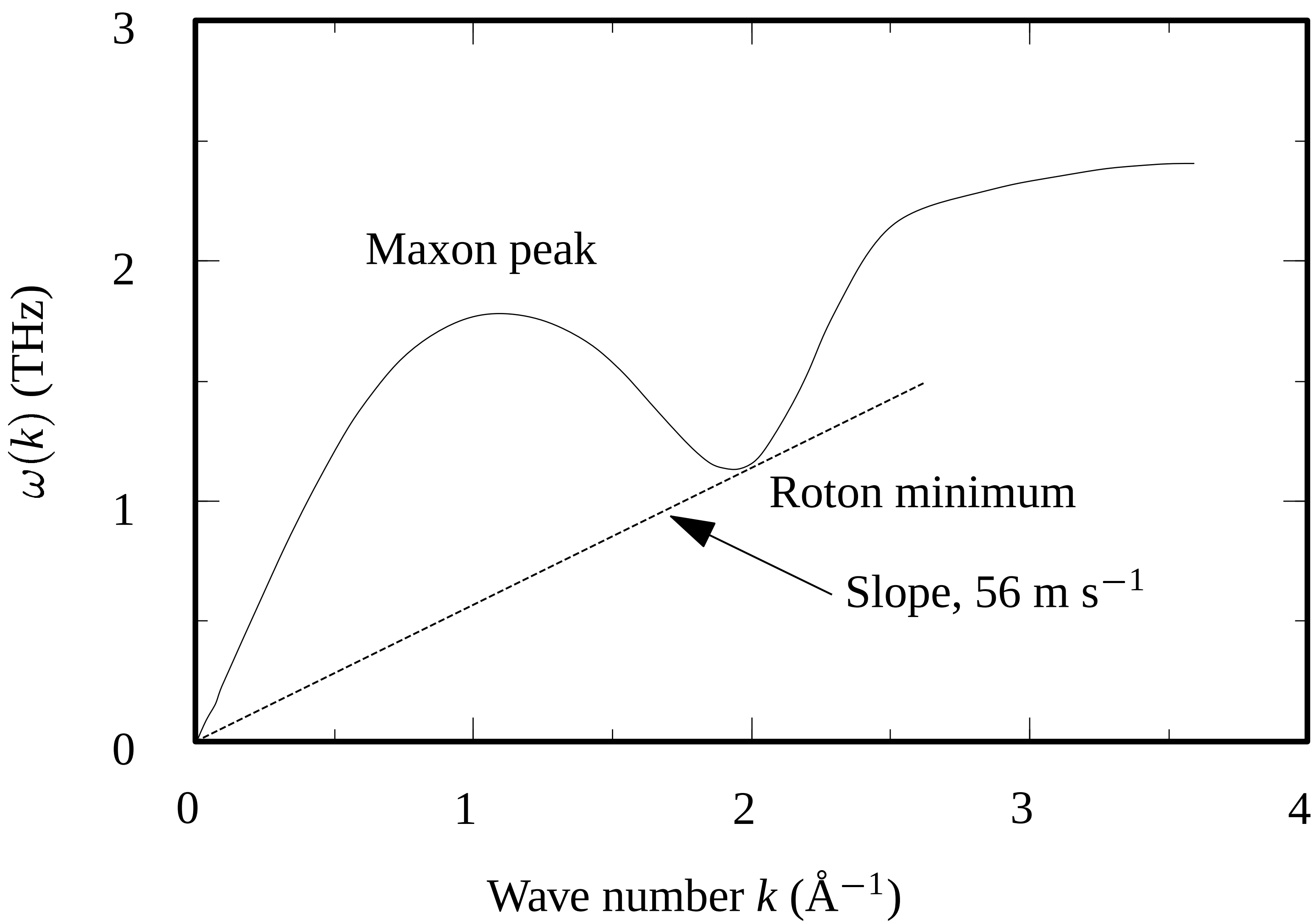



Figure 7: [Source: Donnelly (1991), p. 46.]

# Origin of Superfluidity



$$\Psi(\vec{r}) = \int d^N\vec{r}\,\psi_N^*(\vec{r}_1\ldots\vec{r}_N)\psi_{N+1}(\vec{r}_1\ldots\vec{r}_N,\vec{r}) \tag{L93}$$

$$\hat{\mathcal{H}}_N = \sum_{l=1}^{N}\frac{\hat{P}_l^2}{2m} + U(\vec{r}_1\ldots\vec{r}_N) \tag{L94}$$

$$\frac{\partial\Psi}{\partial t} = \int d^N\vec{r}\,\frac{-i}{\hbar}\left\{\psi_{N+1}\hat{\mathcal{H}}_N\psi_N^* - \psi_N^*\mathcal{H}_{N+1}\psi_{N+1}\right\} \tag{L95}$$

$$= \int d^N\vec{r}\,\frac{-i}{\hbar}\psi_N^*\left\{\frac{-\hbar^2\nabla_{\vec{r}}^2}{2m} + U_{N+1}(\vec{r}_1\ldots\vec{r}_N,\vec{r}) - U_N(\vec{r}_1\ldots\vec{r}_N)\right\}\psi_{N+1}. \tag{L96}$$

$$\frac{-\hbar}{i}\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2\nabla^2}{2m}\Psi + \mu\Psi. \tag{L97}$$

$$\Psi(\vec{r}) = \sqrt{n}e^{i\phi}. \tag{L98}$$

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot \frac{\hbar}{m} \vec{\nabla} \phi n, \tag{L99}$$

$$\vec{v}_s = \frac{\hbar}{m} \vec{\nabla} \phi \tag{L100}$$

$$\hbar \frac{\partial \phi}{\partial t} = -(\mu + m v_s^2/2) + \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}}. \tag{L101}$$

$$m \frac{\partial \vec{v}_s}{\partial t} + m \vec{\nabla} \frac{v_s^2}{2} = -\vec{\nabla} \mu \tag{L102}$$

$$\Rightarrow \frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla}) \vec{v}_s = -\frac{\vec{\nabla} \mu}{m} \tag{L103}$$

$$\int_{\mathcal{C}} d\vec{s} \cdot \vec{v}_s = 2\pi l \hbar / m \tag{L104}$$

$$\Rightarrow \int_{\mathcal{A}} d^2 r \, \hat{z} \cdot \vec{\nabla} \times \vec{v}_s = \kappa = l \frac{h}{m}. \tag{L105}$$

# Origin of Superfluidity

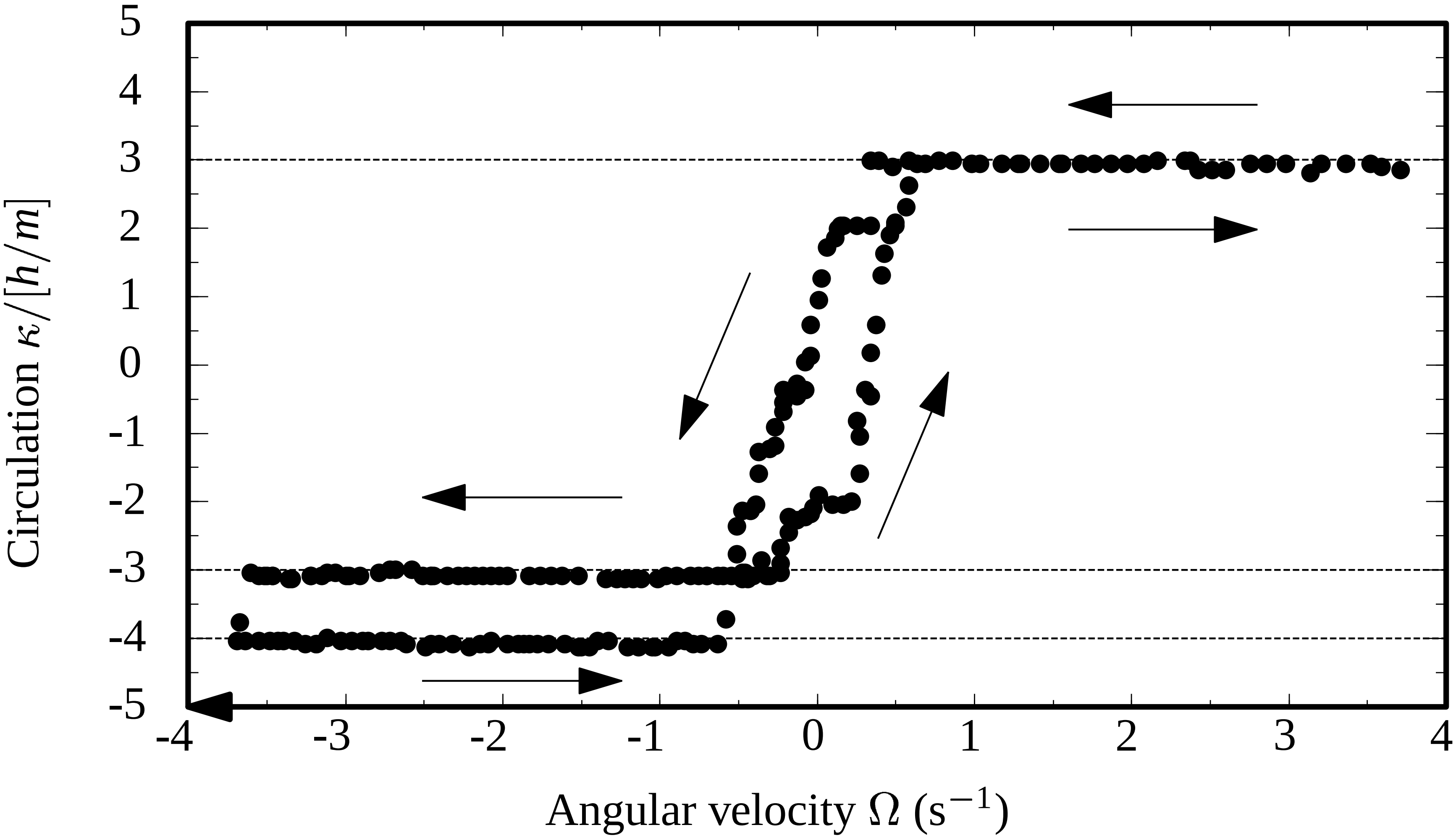



Figure 8: [Source, Karn et al. (1980), p. 1799.]

$$\mathcal{E}_{\rm v} = \int d\vec{r}\, \tfrac{1}{2}\, \rho v^2(\vec{r}) = \tfrac{1}{2}\rho\kappa^2 R(\eta - \tfrac{3}{2}), \quad \eta = \ln(8R/a) \tag{L106}$$

# Origin of Superfluidity



$$P_{\rm v} = \left| \int d\vec{r}\, \rho \vec{v}(\vec{r}) \right| = \rho \kappa \pi R^2 \tag{L107}$$

$$v_{\rm v} = \frac{\partial \mathcal{E}_{\rm v}}{\partial P_{\rm v}} = \frac{\kappa(\eta - \frac{1}{2})}{4\pi R}. \tag{L108}$$

$$\mathcal{E}_{\rm tot} = \mathcal{E}_{\rm v} - P_{\rm v} v_{\rm s} \tag{L109}$$

$$\frac{d\mathcal{E}_{\rm tot}}{dR} = 0 \Rightarrow \frac{\partial \mathcal{E}_{\rm v}}{\partial P_{\rm v}} \frac{\partial P_{\rm v}}{\partial R} - \frac{\partial P_{\rm v}}{\partial R} v_{\rm s} = 0 \quad \Rightarrow v_{\rm s} = v_{\rm v}. \tag{L110}$$

$$i\hbar\frac{\partial}{\partial t}\phi = \left[\sum_l -\frac{\hbar^2\nabla_l^2}{2m} + \hat{U}\right]\phi. \tag{L111}$$

$$\phi\nabla_l\phi^* - \phi^*\nabla_l\phi = 0. \tag{L112}$$

$$\psi(\vec{r}_1\ldots\vec{r}_N) = \exp[\sum_l \Psi(\vec{r}_l,t)]\,\phi(\vec{r}_1\ldots\vec{r}_N) = e^{\sum\Psi_l}\phi. \tag{L113}$$

$$N = 1/\sqrt{\int d^N\vec{r}|\psi|^2} \tag{L114}$$

$$\mathfrak{L} = \int d^N\vec{r}\,\phi^* e^{\sum_{l'}\Psi^*(\vec{r}_{l'})}\left[i\hbar\frac{\partial}{\partial t} + \sum_l \frac{\hbar^2\nabla_l^2}{2m} - \hat{U}\right] e^{\sum_{l''}\Psi(\vec{r}_{l''})}\phi. \tag{L115}$$

$$\int e^{\sum\Psi^*_{l'}}\phi^*\,\nabla_l^2\, e^{\sum\Psi_{l''}}\phi \tag{L116}$$

$$= \int e^{\sum \Psi^*_{l'}} \phi^* \left[ \phi \nabla_l^2 e^{\sum \Psi_{l''}} + 2 \vec{(} \nabla_l e^{\sum \Psi_{l''}}) \cdot (\vec{\nabla}_l \phi) + e^{\sum \Psi_{l''}} \nabla_l^2 \phi \right] \quad \text{(L117)}$$

$$= \int e^{\sum \Psi^*_{l'}} \left[ |\phi|^2 \nabla_l^2 e^{\sum \Psi_{l''}} + (\vec{\nabla}_l e^{\sum \Psi_{l''}}) \cdot (\vec{\nabla}_l |\phi|^2) + e^{\sum \Psi_{l''}} \phi^* \nabla_l^2 \phi \right] \quad \text{(L118)}$$

$$= \int e^{\sum \Psi^*_{l'}} |\phi|^2 \nabla_l^2 e^{\sum \Psi_{l'}} - |\phi|^2 \vec{\nabla}_l \cdot e^{\sum \Psi^*_{l'}} (\vec{\nabla}_l e^{\sum \Psi_{l''}}) + e^{\sum \Psi^*_{l'} + \Psi_{l'}} \phi^* \nabla_l^2 \phi \quad \text{(L119)}$$

$$= \int e^{\sum \Psi^*_{l'} + \Psi_{l'}} \left[ \phi^* \nabla_l^2 \phi - |\phi|^2 |\nabla_l \Psi_l|^2 \right]. \quad \text{(L120)}$$

$$\mathfrak{L} = \int d^N \vec{r} |\phi|^2 e^{\sum_{l'} \Psi^*_{l'} + \Psi_{l'}} \sum_l \left[ i\hbar \frac{\partial \Psi_l}{\partial t} - \frac{\hbar^2}{2m} |\vec{\nabla}_l \Psi_l|^2 \right]. \quad \text{(L121)}$$

$$\frac{\delta}{\delta \Psi^*} \left[ \mathfrak{L} - \mu \int d^N \vec{r} |\phi^2| e^{\sum_{l'} \Psi^*_{l'} + \Psi_{l'}} \right] = 0. \quad \text{(L122)}$$

$$n_1(\vec{r}) = \int d^N\vec{r}\,|\phi|^2 e^{\sum_{l'} \Psi^*_{l'} + \Psi_{l'}} \sum_l \delta(\vec{r} - \vec{r}_l) \qquad \text{(L123)}$$

$$S(\vec{r},\vec{r}') = \frac{\mathcal{V}}{N} \int d^N\vec{r}\,|\phi|^2 e^{\sum_{l'} \Psi^*_{l'} + \Psi_{l'}} \sum_{ll'} \delta(\vec{r} - \vec{r}_l)\delta(\vec{r}' - \vec{r}_{l'}). \qquad \text{(L124)}$$

$$-\frac{\hbar^2}{2m}\vec{\nabla}\cdot n_1\vec{\nabla}\Psi(\vec{r}) = \sum_{ll'} \int d^N\vec{r}|\psi|^2\delta(\vec{r}_{l'} - \vec{r})\left[i\hbar\frac{\partial \Psi_l}{\partial t} - \frac{\hbar^2}{2m}|\nabla\Psi_l|^2 - \mu\right] \qquad \text{(L125)}$$

$$= \frac{N}{\mathcal{V}} \int d\vec{r}'\left[i\hbar\frac{\partial \Psi(\vec{r}')}{\partial t} - \frac{\hbar^2}{2m}|\nabla\Psi(\vec{r}')|^2 - \mu\right] S(\vec{r},\vec{r}'). \qquad \text{(L126)}$$

$$\frac{\hbar^2 k^2}{2m}\Psi(\vec{q},\omega) = [\hbar\omega\Psi(\vec{q},\omega) - \mu\delta(\vec{q})\delta(\omega)]S(\vec{q}) \qquad \text{(L127)}$$

$$\Rightarrow \hbar\omega(\vec{q}) = \frac{\hbar^2 q^2}{2mS(\vec{q})} = \frac{6.02k_B[q\cdot\text{Å}]^2}{S(q)}\text{K}, \qquad \text{(L128)}$$

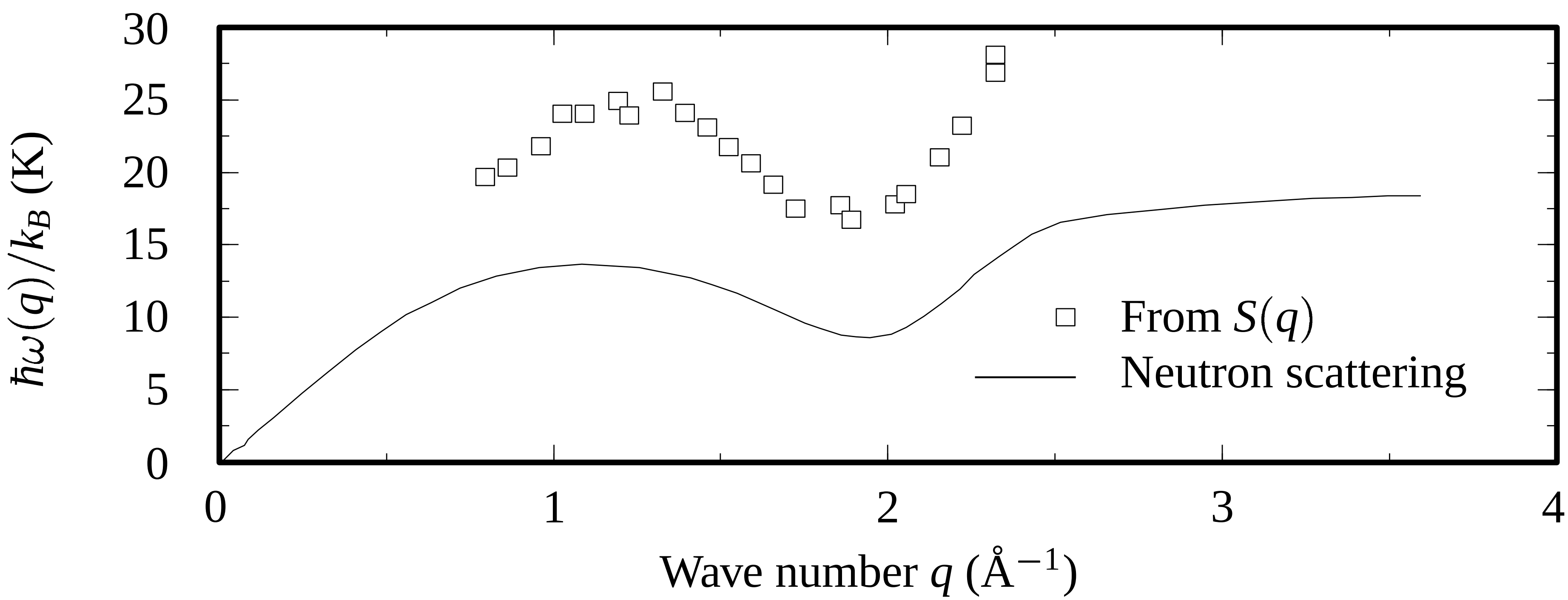



Figure 9: The neutron scattering data are from Donnelly (1991) p. 46, and data for $S(q)$ are from Svensson et al. (1980).

# Superfluid $^3$He

Need something....