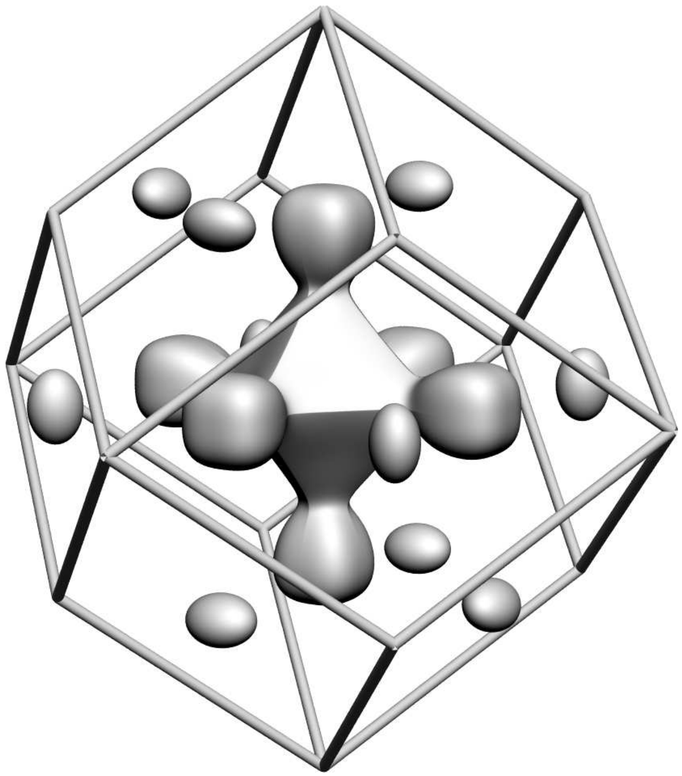
Dynamics of Bloch Electrons



Definitions

- ☞ Drude model
- ☞ Semiclassical dynamics
- ☞ Bloch oscillations
- ☞ $\vec{K} \cdot \vec{P}$ method
- ☞ Effective mass
- ☞ Houston states
- ☞ Zener tunneling
- ☞ Wave packets
- ☞ Anomalous velocity
- ☞ Wannier–Stark ladders
- ☞ de Haas–van Alphen effect

$$m\dot{\vec{v}} = -e\vec{E} - e\frac{\vec{v}}{c} \times \vec{B} - m\frac{\vec{v}}{\tau}, \tag{L1}$$

In the absence of an electric field,

$$\vec{v}(t) = \vec{v}_0 e^{-t/\tau}. \tag{L2}$$

In the presence of one

$$\vec{v}(t) = ? \qquad ? + [\vec{v}_0 + \frac{\tau e}{m}\vec{E}]e^{-t/\tau} \tag{L3}$$

$$\vec{v} = ? \qquad ? \tag{L4}$$

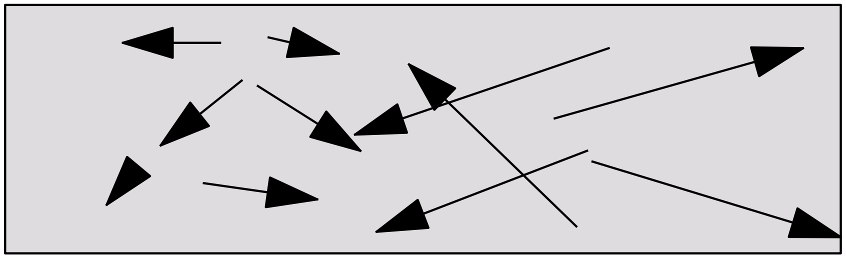
$$\vec{j} = -ne\vec{v} = \frac{ne^2\tau}{m}\vec{E} \tag{L5}$$

$$\Rightarrow \sigma = \frac{ne^2\tau}{m}, \tag{L6}$$

Drude Model

$$\tau = \frac{m}{ne^2\rho} = \frac{3.55\cdot 10^{-13}\,\mathrm{s}}{n/[10^{22}\,\mathrm{cm}^{-3}]\;\rho/[\mu\Omega\,\mathrm{cm}]} \qquad \text{(L7)}$$

Colder Hotter



$x - v\tau$ x $x + v\tau$

$$j_{\mathcal{E}} = \frac{n}{2}v_x\left[\mathcal{E}(x - v_x\tau) - \mathcal{E}(x + v_x\tau)\right] \approx -nv_x^2\tau\frac{\partial\mathcal{E}}{\partial x} = -nv_x^2\tau\frac{\partial\mathcal{E}}{\partial T}\frac{\partial T}{\partial x} \qquad \text{(L8)}$$

$$= -\frac{2n}{m}\frac{1}{2}mv_x^2c_V\tau\frac{\partial T}{\partial x} = -\frac{\tau n}{m}\frac{3k_B^2T}{2}\frac{\partial T}{\partial x} \qquad \text{(L9)}$$

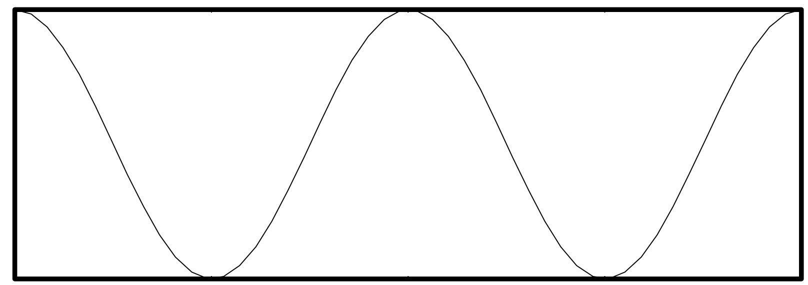
$$\Rightarrow \frac{\kappa}{\sigma T} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 = 1.24\cdot 10^{-13}\mathrm{erg\,cm^{-1}\,K^{-2}}. \qquad \text{(L10)}$$

Semiclassical Electron Dynamics

$$\dot{\vec{r}} = \frac{1}{\hbar}\frac{\partial \mathcal{E}_{\vec{k}}}{\partial \vec{k}}. \tag{L11}$$

$$\hbar\dot{\vec{k}} = -e\vec{E} - \frac{e}{c}\dot{\vec{r}} \times \vec{B} \tag{L12}$$

Bloch Oscillations



$$\mathcal{E}_k = -2\mathfrak{t}\cos ak, \tag{L13}$$

$$\hbar\dot{k} = -eE \tag{L14}$$

$$\Rightarrow k = -eEt/\hbar \tag{L15}$$

$$\Rightarrow \dot{r} = -\frac{2\mathfrak{t}a}{\hbar}\sin\left(\frac{aeEt}{\hbar}\right) \tag{L16}$$

$$\Rightarrow r = \frac{2\mathfrak{t}}{eE}\cos\left(\frac{aeEt}{\hbar}\right). \tag{L17}$$

Bloch Oscillations

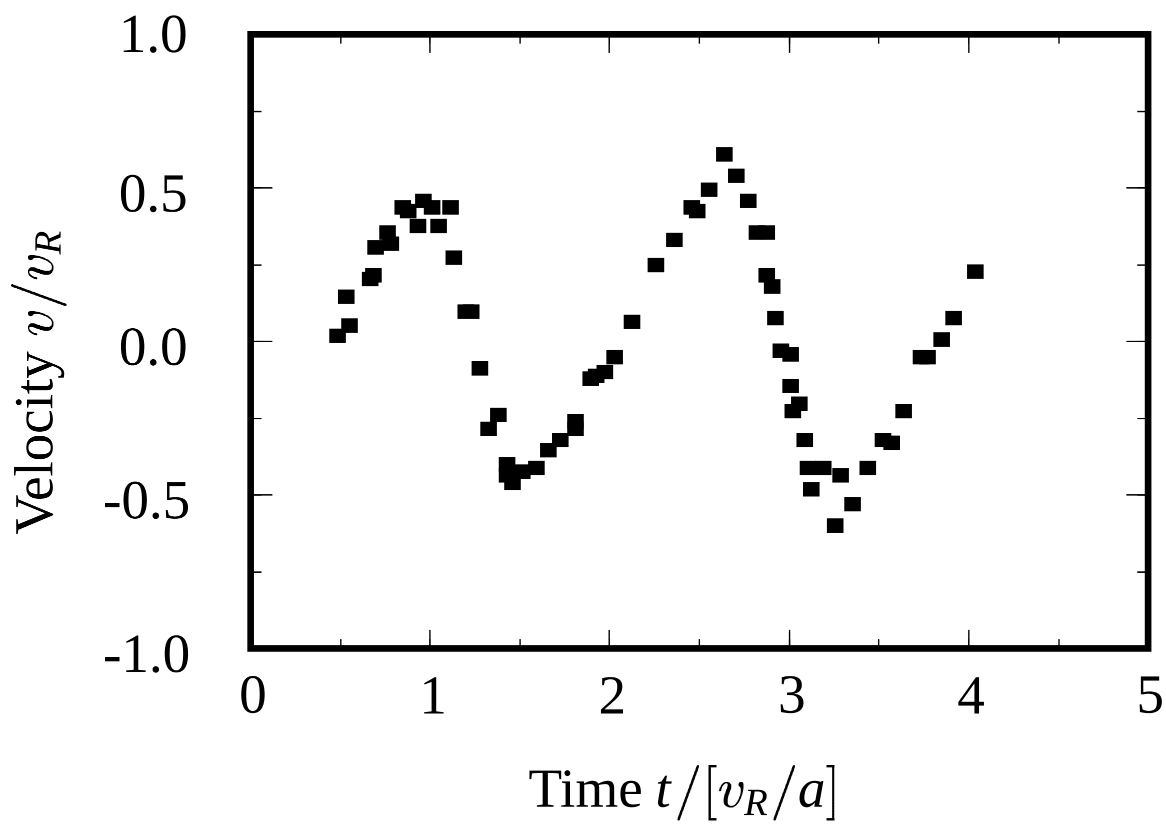


Figure 1: [Source: ben Dahan et al. (1996), p. 4510.]

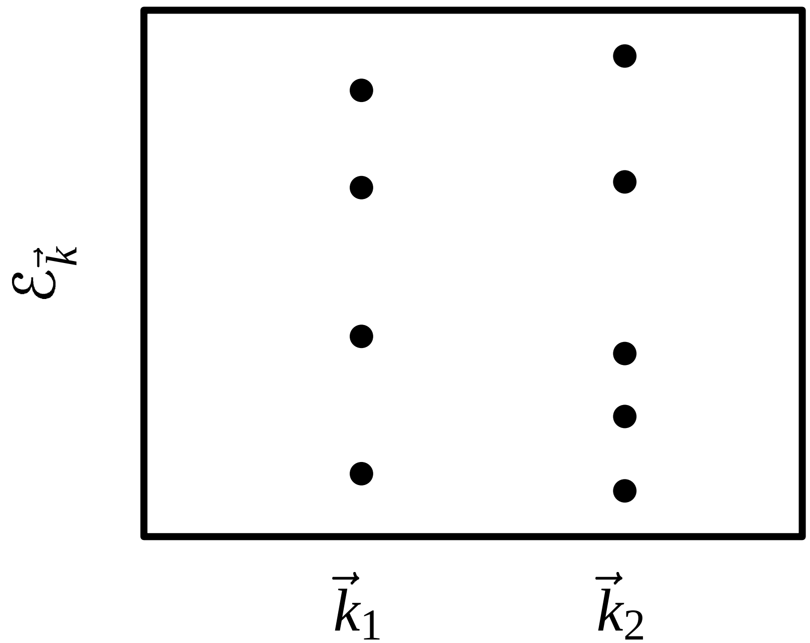


Figure 2: Which eigenvalues belong to the same band?

$$\hat{\mathcal{H}}_{\vec{k}+\vec{\delta k}} = \frac{\hbar^2}{2m}[-\vec{\nabla}^2 - 2i(\vec{k}+\vec{\delta}k)\cdot\vec{\nabla} + |\vec{k}+\vec{\delta}k|^2]u(\vec{r}) + U(\vec{r})u(\vec{r}) = \mathcal{E}u(\vec{r}). \quad \text{(L18)}$$

$$\hat{\mathcal{H}}^{(1)}_{\vec{k}} = -\frac{\hbar^2}{2m}\left[-\delta k^2 - 2\vec{\delta}k\cdot\vec{k} + 2i\vec{\delta}k\cdot\vec{\nabla}\right]. \quad \text{(L19)}$$

$$\mathcal{E}_{n,\vec{k}+\vec{\delta}k} = \mathcal{E}_{n\vec{k}} + \mathcal{E}^{(1)}_{n\vec{k}} + \mathcal{E}^{(2)}_{n\vec{k}} + \ldots. \tag{L20}$$

$$\mathcal{E}^{(1)}_{n\vec{k}} = \langle u_{n\vec{k}}|(\frac{\hbar^2}{m})\vec{\delta}k \cdot (\vec{k} - i\vec{\nabla})|u_{n\vec{k}}\rangle. \tag{L21}$$

$$(\vec{k} - i\vec{\nabla})e^{-i\vec{k}\cdot\vec{r}} = -ie^{-i\vec{k}\cdot\vec{r}}\vec{\nabla}. \tag{L22}$$

$$\mathcal{E}^{(1)}_{n\vec{k}} = \frac{\hbar}{m}\langle\psi_{n\vec{k}}|\vec{\delta}k \cdot \hat{P}|\psi_{n\vec{k}}\rangle \tag{L23}$$

$$\Rightarrow \frac{\partial \mathcal{E}_{n\vec{k}}}{\partial \vec{k}} = \frac{\hbar}{m}\langle\psi_{n\vec{k}}|\hat{P}|\psi_{n\vec{k}}\rangle \tag{L24}$$

$$\Rightarrow \frac{\partial \mathcal{E}_{n\vec{k}}}{\partial \hbar\vec{k}} = \langle\hat{v}\rangle \equiv \vec{v}_{n\vec{k}}. \tag{L25}$$

Effective Mass

$$\frac{d}{dt}\langle\hat{v}_\alpha\rangle = \sum_\beta \frac{\partial\langle\hat{v}_\alpha\rangle}{\partial k_\beta}\frac{\partial k_\beta}{\partial t} \tag{L26}$$

$$\Rightarrow \frac{d}{dt}\langle\hat{v}\rangle = \hbar\mathbf{M}^{-1}\dot{\vec{k}}, \tag{L27}$$

where

$$(\mathbf{M}^{-1})_{\alpha\beta} \equiv \frac{1}{\hbar^2}\frac{\partial^2\mathcal{E}_{n\vec{k}}}{\partial k_\alpha \partial k_\beta}. \tag{L28}$$

Proceeding to second order....

$$(\mathbf{M}^{-1})_{\alpha\beta} = \frac{1}{m}\delta_{\alpha\beta} + \frac{1}{m^2}\sum_{n'\neq n}\frac{\langle\psi_{n\vec{k}}|\hat{P}_\alpha|\psi_{n'\vec{k}}\rangle\langle\psi_{n'\vec{k}}|\hat{P}_\beta|\psi_{n\vec{k}}\rangle + \text{c.c.}}{\mathcal{E}_{n\vec{k}} - \mathcal{E}_{n'\vec{k}}} \tag{L29}$$

Potential of form $-\vec{E}\cdot\vec{r}$ conflicts with periodic boundary conditions.

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V. \tag{L30}$$

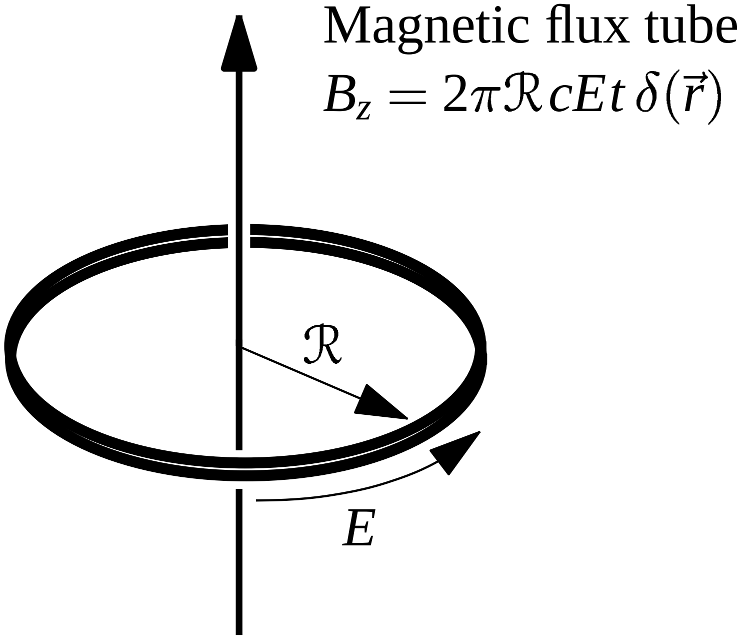


Figure 3: A thin tube of increasing magnetic flux through a loop of wire.

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{P} + \frac{e}{c}A\right)^2 + \hat{U}(\hat{R}), \tag{L31}$$

Electrons in Electric Field

$$A = -cEt. \tag{L32}$$

$$\left[\frac{1}{2m}\left(\hat{P} + \frac{e}{c}A\right)^2 + \hat{U}\right]\tilde{\phi}(x,t) = \mathcal{E}_t\tilde{\phi}(x,t). \tag{L33}$$

$$\tilde{\phi}(x+L) = \tilde{\phi}(x). \tag{L34}$$

$$\tilde{\phi}(x,t) = e^{-ieAx/\hbar c}\phi(x,t). \tag{L35}$$

$$[\frac{\hat{P}^2}{2m} + \hat{U}]\phi(x,t) = \mathcal{E}_t\phi(x,t). \tag{L36}$$

$$\phi_{nk(t)}(x) = e^{ik(t)x}u_{nk(t)}(x). \tag{L37}$$

$$e^{-ieA(x+L)/\hbar c}e^{ik(t)(x+L)}u_{nk(t)}(x+L) \quad = \quad ? \qquad ? \tag{L38}$$

$$\Rightarrow \frac{-eA}{\hbar c} + k(t) = \frac{2\pi l}{L}. \tag{L39}$$

$$\Rightarrow \frac{eEt}{\hbar} + k(t) = \frac{2\pi l}{L}. \tag{L40}$$

(L41)

$$\hbar\dot{k} = -eE. \tag{L42}$$

Zener Tunneling

$$\exp\left[\frac{i}{\hbar}\int_0^x dx'\,\sqrt{2m(-\mathcal{E}_g)}\right] \tag{L43}$$

$$\sim \exp\left[-x\sqrt{\frac{2m\mathcal{E}_g}{\hbar^2}}\right] \tag{L44}$$

$$\sim \exp\left[-\frac{\mathcal{E}_g}{eE}\sqrt{\frac{2m\mathcal{E}_g}{\hbar^2}}\right]. \tag{L45}$$

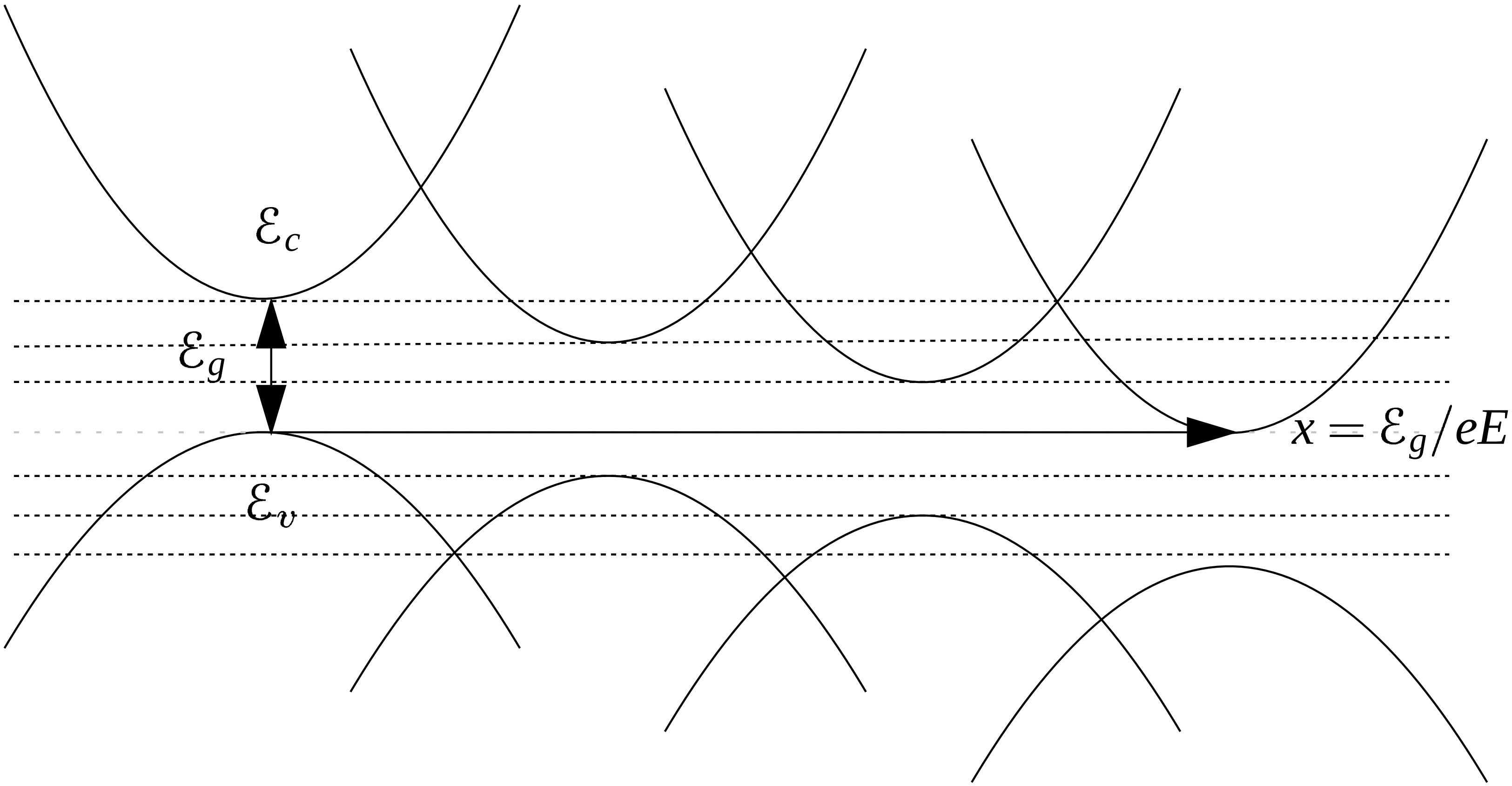


Figure 4: Energy diagram of Zener tunneling.

$$|\psi(t)\rangle = \sum_{n'} C_{n'}(t)|\tilde{\phi}_{n'k(t)}\rangle. \quad \text{(L46)}$$

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = \hat{\mathcal{H}}|\psi\rangle \tag{L47}$$

$$\hat{\mathcal{H}}|\psi\rangle = \sum_{n'} C_{n'}(t)\mathcal{E}_{n'k(t)}|\tilde{\phi}_{n'k(t))}\rangle \tag{L48}$$

$$= i\hbar \sum_{n'} \frac{\partial C_{n'}}{\partial t}|\tilde{\phi}_{n'k(t)}\rangle + C_{n'}(t)\frac{\partial}{\partial k}|\tilde{\phi}_{n'k(t)}\rangle \dot{k} \tag{L49}$$

$$\Rightarrow \langle\tilde{\phi}_{nk(t)}|\hat{\mathcal{H}}|\psi\rangle = C_n(t)\mathcal{E}_{nk(t)} \tag{L50}$$

$$= i\hbar \frac{\partial C_n}{\partial t} - \sum_{n'} iC_{n'}\langle\tilde{\phi}_{nk(t)}|\frac{\partial\tilde{\phi}_{n'k(t)}}{\partial k}\rangle eE. \tag{L51}$$

$$C_1\mathcal{E}_{1k(t)} = i\hbar \frac{\partial C_1}{\partial t} \tag{L52}$$

$$\Rightarrow C_1 = \exp\left[-\frac{i}{\hbar}\int_0^t dt'\,\mathcal{E}_{1k(t')}\right]. \tag{L53}$$

Zener Tunneling

$$\alpha_2(t) = C_2(t)\exp\left[\frac{i}{\hbar}\int_0^t dt'\mathcal{E}_{2k(t')}\right], \tag{L54}$$

$$\dot{\alpha}_2 = \langle\tilde{\phi}_{2k(t)}|\frac{\partial\tilde{\phi}_{1k(t)}}{\partial k}\rangle\frac{eE}{\hbar}\exp\left[\frac{i}{\hbar}\int_0^t dt'(\mathcal{E}_{2k(t')}-\mathcal{E}_{1k(t')})\right]. \tag{L55}$$

$$\alpha_2(\mathfrak{T}) \approx \frac{L}{N}\int_0^{\mathfrak{T}} dt\,\frac{eE}{\hbar}\exp\left[\frac{i}{\hbar}\int_0^t dt'(\mathcal{E}_{2k(t')}-\mathcal{E}_{1k(t')})\right]. \tag{L56}$$

$$\alpha_2(\mathfrak{T}) \approx \frac{L}{N}\int_0^{2\pi N/L} dk\,\exp\left[\frac{-i}{eE}\int_0^k dk'(\mathcal{E}_{2k'}-\mathcal{E}_{1k'})\right]. \tag{L57}$$

$$\frac{1}{m^\star} = \left[\frac{1}{m_v^\star}+\frac{1}{m_c^\star}\right] \tag{L58}$$

gives

$$\mathcal{E}_{2k'}-\mathcal{E}_{1k'} = \mathcal{E}_g+\frac{\hbar^2k'^2}{2m^\star}. \tag{L59}$$

$$\mathcal{E}_g + \frac{\hbar^2 q^2}{2m^\star} = 0. \tag{L60}$$

$$\alpha_2(\mathcal{T}) \sim \exp\left[\frac{-i}{eE}\int_0^q dk'\,\mathcal{E}_g + \frac{\hbar^2 k'^2}{2m^\star}\right]. \tag{L61}$$

$$\sim \exp\left[\frac{-2i}{3eE} q\mathcal{E}_g\right] \tag{L62}$$

$$\sim \exp\left[\frac{-2\mathcal{E}_g^{3/2}}{3eE}\sqrt{\frac{2m^\star}{\hbar^2}}\right] \tag{L63}$$

$$\sim \exp\left[-3.41\cdot 10^7\,[\mathcal{E}_g/\mathrm{eV}]^{3/2}[m^\star/m]^{1/2}/[E\cdot\mathrm{cm\,V}^{-1}]\right]. \tag{L64}$$

$$W_{\vec{r}_c\vec{k}_c}(\vec{r}) = \frac{1}{\sqrt{N}}\sum_{\vec{k}} w_{\vec{k}\vec{k}_c} e^{-ie\vec{A}(\vec{r}_c)\cdot\vec{r}/\hbar c - i\vec{k}\cdot\vec{r}_c}\psi_{\vec{k}}(\vec{r}). \tag{L65}$$

Calculations from here on out too complex to present at board...

$$1 = \langle W_{\vec{r}_c\vec{k}_c}|W_{\vec{r}_c\vec{k}_c}\rangle = \frac{1}{N}\sum_{\vec{k}\vec{k}'}\int d\vec{r}\, e^{i(\vec{k}'-\vec{k})\cdot\vec{r}_c} w_{\vec{k}\vec{k}_c} w^*_{\vec{k}'\vec{k}_c}\psi^*_{\vec{k}'}(\vec{r})\psi_{\vec{k}}(\vec{r}) = \sum_{\vec{k}\vec{k}'} w_{\vec{k}\vec{k}_c} w^*_{\vec{k}'\vec{k}_c}\delta_{\vec{k}\vec{k}'} \tag{L66}$$

$$\Rightarrow 1 \quad = \quad \sum_{\vec{k}} |w_{\vec{k}\vec{k}_c}|^2. \tag{L67}$$

Formal Dynamics of Wave Packets

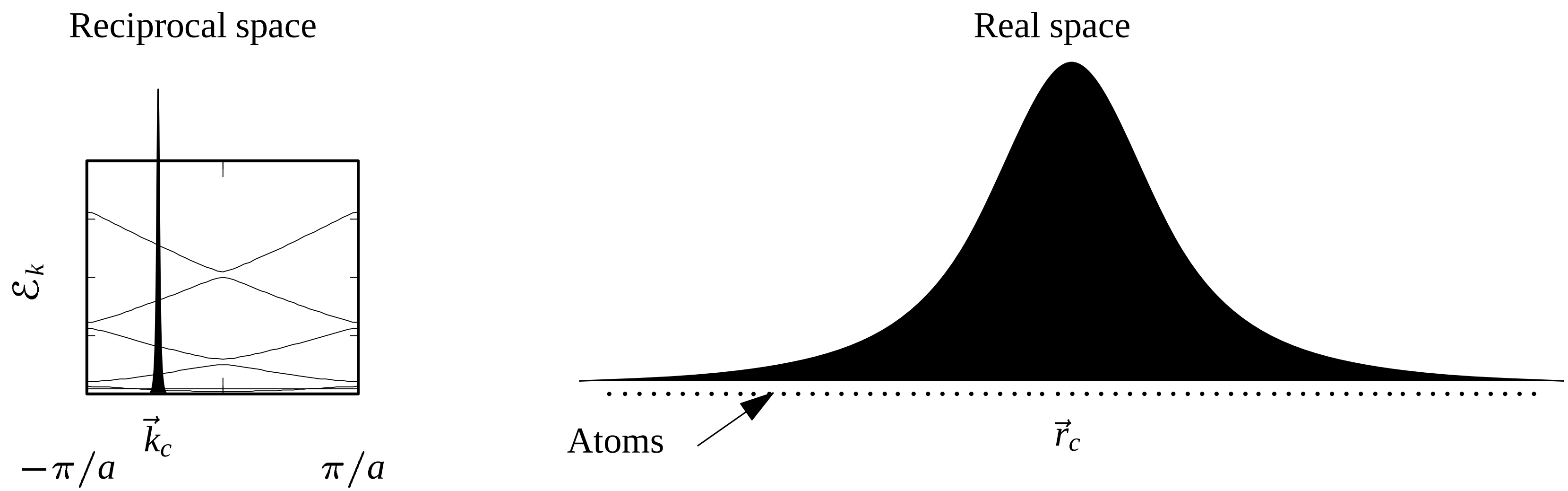


Figure 5: A wave packet viewed in real and reciprocal space

$$w_{\vec{k}\vec{k}_c} = |w|_{\vec{k}-\vec{k}_c} e^{i(\vec{k}-\vec{k}_c)\cdot\vec{\mathcal{R}}_{\vec{k}_c}}, \tag{L68}$$

where

$$\vec{\mathcal{R}}_{\vec{k}_c} = i\int_\Omega d\vec{r}\, u^*_{\vec{k}_c}(\vec{r})\frac{\partial}{\partial\vec{k}_c}u_{\vec{k}_c}(\vec{r}). \tag{L69}$$

$$\langle W_{\vec{r}_c\vec{k}_c}|\vec{r}-\vec{r}_c|W_{\vec{r}_c\vec{k}_c}\rangle = \langle W_{\vec{r}_c\vec{k}_c}|\vec{r}|W_{\vec{r}_c\vec{k}_c}\rangle - r_c \tag{L70}$$

$$= \int \frac{d\vec{r}}{N} \sum_{\vec{k}\vec{k}'} w^*_{\vec{k}\vec{k}_c} w_{\vec{k}'\vec{k}_c} e^{i(\vec{k}'-\vec{k})\cdot(\vec{r}-\vec{r}_c)} u^*_{\vec{k}}(\vec{r}) u_{\vec{k}'}(\vec{r}) \left[\vec{r}-\vec{r}_c\right] \quad \text{(L71)}$$

(L72)

$$= \int \frac{d\vec{r}}{N} \sum_{\vec{k}'\vec{k}} w^*_{\vec{k}\vec{k}_c} w_{\vec{k}'\vec{k}_c} u^*_{\vec{k}}(\vec{r}) u_{\vec{k}'}(\vec{r}) \frac{\partial}{\partial i\vec{k}'} e^{i(\vec{k}'-\vec{k})\cdot(\vec{r}-\vec{r}_c)} \quad \text{(L73)}$$

$$= -\int \frac{d\vec{r}}{N} \sum_{\vec{k}'\vec{k}} w^*_{\vec{k}\vec{k}_c} u^*_{\vec{k}}(\vec{r}) e^{i(\vec{k}'-\vec{k})\cdot(\vec{r}-\vec{r}_c)} \frac{\partial}{\partial i\vec{k}'} \left[w_{\vec{k}'\vec{k}_c} u_{\vec{k}'}(\vec{r})\right] \quad \text{(L74)}$$

$$= -\int_\Omega d\vec{r} \sum_{\vec{k}'\vec{k}} \delta_{\vec{k}\vec{k}'} w^*_{\vec{k}\vec{k}_c} u^*_{\vec{k}}(\vec{r}) \frac{\partial}{\partial i\vec{k}'} \left[w_{\vec{k}'\vec{k}_c} u_{\vec{k}'}(\vec{r})\right] \quad \text{(L75)}$$

$$= -\int_\Omega d\vec{r} \sum_{\vec{k}} |w|^2_{\vec{k}-\vec{k}_c} u^*_{\vec{k}}(\vec{r}) \frac{1}{w_{\vec{k}\vec{k}_c}} \frac{\partial}{\partial i\vec{k}} \left[w_{\vec{k}\vec{k}_c} u_{\vec{k}}(\vec{r})\right] \quad \text{(L76)}$$

$$= \int_\Omega d\vec{r}\, i u^*_{\vec{k}_c}(\vec{r}) \frac{\partial}{\partial \vec{k}_c} u_{\vec{k}_c}(\vec{r}) - \frac{\partial}{\partial i\vec{k}} \ln w_{\vec{k}\vec{k}_c} \Big|_{\vec{k}=\vec{k}_c} = 0 \quad \text{(L77)}$$

$$\Rightarrow \quad \langle W_{\vec{r}_c\vec{k}_c}|\vec{r}|W_{\vec{r}_c\vec{k}_c}\rangle = \vec{r}_c \tag{L78}$$

$$\mathcal{L} = \langle W_{\vec{r}_c\vec{k}_c}|i\hbar\frac{\partial}{\partial t}|W_{\vec{r}_c\vec{k}_c}\rangle - \langle W_{\vec{r}_c\vec{k}_c}|\hat{\mathcal{H}} - eV(\vec{r})|W_{\vec{r}_c\vec{k}_c}\rangle \tag{L79}$$

$$\hat{\mathcal{H}} = \frac{1}{2m}[\hat{P} + \frac{e\vec{A}(\vec{r})}{c}]^2 + U(\vec{r}) \tag{L80}$$

$$[\frac{\hat{P}^2}{2m} + U(\vec{r})]\psi_{\vec{k}} = \mathcal{E}_{\vec{k}}\psi_{\vec{k}}. \tag{L81}$$

$$\langle W_{\vec{r}_c\vec{k}_c}|i\hbar\frac{\partial}{\partial t}|W_{\vec{r}_c\vec{k}_c}\rangle = \frac{e\vec{r}_c}{c}\cdot\frac{d\vec{A}(\vec{r}_c)}{dt} + \hbar\vec{k}_c\cdot\dot{\vec{r}}_c + \hbar\dot{\vec{k}}_c\cdot\vec{\mathcal{R}}_{\vec{k}_c} \tag{L82a}$$

$$\langle W_{\vec{r}_c\vec{k}_c}|\hat{\mathcal{H}} - eV(\vec{r})|W_{\vec{r}_c\vec{k}_c}\rangle = \mathcal{E}_{\vec{k}_c} + \frac{e}{2mc}\vec{B}\cdot\vec{L}_{\vec{k}_c} - eV(\vec{r}_c) \tag{L82b}$$

with

$$\vec{L}_{\vec{k}_c} = \frac{\hbar}{2}\int_\Omega d\vec{r}\,[\frac{\partial u_{\vec{k}_c}}{\partial i\vec{k}_c}^* - \vec{\mathcal{R}}_{\vec{k}_c}u^*_{\vec{k}_c}]\times[\frac{\partial}{\partial i\vec{r}} + \vec{k}_c]u_{\vec{k}_c} + \text{c.c.}. \tag{L82c}$$

$$\frac{\partial \mathcal{L}}{\partial \vec{r}_c} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_c} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \vec{k}_c} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{k}}_c}. \tag{L83}$$

$$\hbar\dot{\vec{k}}_c = -e\vec{E} - \frac{e}{c}\dot{\vec{r}}_c \times \vec{B} \tag{L84a}$$

$$\dot{\vec{r}}_c = \frac{1}{\hbar}\left[\frac{\partial \mathcal{E}_{\vec{k}_c}}{\partial \vec{k}_c} + \frac{e}{2mc}\vec{B}\cdot\frac{\partial \vec{L}_{\vec{k}_c}}{\partial \vec{k}_c}\right] - \dot{\vec{k}}_c \times \vec{\Omega}, \tag{L84b}$$

Recover expected semiclassical dynamics, but with corrections due to anomalous velocity $\vec{\Omega}$.

$$\vec{B}(\vec{r}) = \frac{\partial}{\partial \vec{r}} \times \vec{A}(\vec{r}) \tag{L85a}$$

$$\vec{\Omega}(\vec{k}) = \frac{\partial}{\partial \vec{k}} \times \vec{\mathcal{R}}(\vec{k}). \tag{L85b}$$

Conditions for validity of Semiclassical Dynamics

$$\frac{eE}{k_F} \ll \mathcal{E}_g \sqrt{\frac{\mathcal{E}_g}{\mathcal{E}_F}}. \tag{L86}$$

$$2\pi\hbar/\mathfrak{T} \ll \mathcal{E}_g \sqrt{\frac{\mathcal{E}_g}{\mathcal{E}_F}}. \tag{L87}$$

$$\mathcal{H} = \sum_l \dot{Q}_l P_l - \mathcal{L}; \quad P_l = \frac{\partial \mathcal{L}}{\partial \dot{Q}_l}. \tag{L88}$$

$$\vec{p} \quad = \quad \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \hbar \vec{k} - \frac{e\vec{A}}{c} \Rightarrow \hbar \vec{k} = \vec{p} + e\vec{A}/c \tag{L89a}$$

$$\vec{\pi} \quad = \quad \frac{\partial \mathcal{L}}{\partial \dot{\vec{k}}} = \hbar \vec{\mathcal{R}}_{\vec{k}}, \tag{L89b}$$

$$\mathcal{H} = \mathcal{E}_{\vec{k}} - eV(\vec{r}) + (e/2mc)\vec{B} \cdot \vec{L}_{\vec{k}} \equiv \mathcal{E}(\vec{p} + e\vec{A}/c) - eV(\vec{r}) + (e/2mc)\vec{B} \cdot \vec{L}_{\vec{k}}. \tag{L90}$$

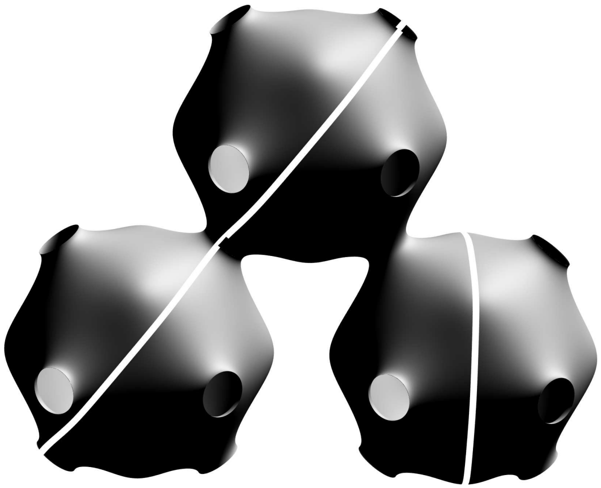


Figure 6: Energy contours on the Fermi surface of copper, showing open and closed orbits.

$$i\hbar\frac{\partial}{\partial t}|W\rangle = \hat{\mathcal{H}}|W\rangle. \tag{L91}$$

$$i\hbar\frac{\partial}{\partial t}|W\rangle = \mathcal{H}|W\rangle, \tag{L92}$$

$$e^{-i\mathcal{H}t/\hbar}. \tag{L93}$$

$$\mathcal{H}\mathcal{T} = 2\pi\hbar j, \tag{L94}$$

$$2\pi\hbar j = \int dt \sum_l P_l \frac{\partial\mathcal{H}}{\partial P_l} = \oint \sum_l dQ_l P_l, \tag{L95}$$

$$\oint \quad d\vec{k} \quad \cdot\vec{\mathcal{R}}_{\vec{k}} + d\vec{r}\cdot[\vec{k} - \frac{e\vec{A}}{\hbar c}] = 2\pi j \tag{L96}$$

$$\Rightarrow \oint \quad d\vec{k} \quad \cdot(\vec{\mathcal{R}}_{\vec{k}} - \vec{r}) - d\vec{r}\cdot\frac{e\vec{A}}{\hbar c} = 2\pi j. \tag{L97}$$

$$\Gamma = \oint d\vec{k} \cdot \vec{\mathcal{R}}_{\vec{k}}, \tag{L98}$$

$$2\pi j = \oint d\vec{k} \cdot (\vec{\mathcal{R}}_{\vec{k}} - \vec{r}) = \Gamma - \int_0^K d\vec{k} \cdot \vec{r} = \Gamma - K \langle \vec{r} \rangle \tag{L99}$$

$$\Rightarrow \langle \vec{r} \rangle = \frac{\Gamma - 2\pi j}{K}. \tag{L100}$$

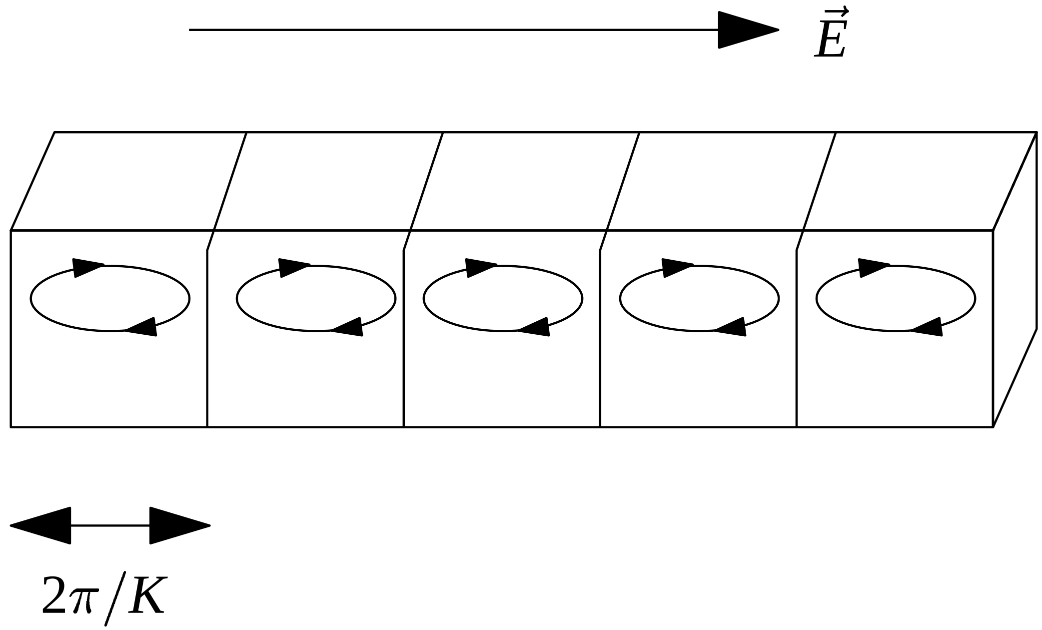


Figure 7: The Wannier–Stark ladder is a collection of electrons trapped in Bloch oscillations by an intense electric field, and spaced at intervals of $2\pi/K$, where $\vec{K}$ is a reciprocal lattice vector.

de Haas–van Alphen Effect

$$\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}, \tag{L101}$$

$$\dot{\vec{k}} = \frac{-e\dot{\vec{r}}}{\hbar c} \times \vec{B} \quad \Rightarrow \quad \vec{k}(t) - \vec{k}(0) = \frac{-e}{\hbar c}[\vec{r}(t) - \vec{r}(0)] \times \vec{B} \tag{L102}$$

$$\Rightarrow \vec{B} \times (\vec{k}(t) - \vec{k}(0)) = \frac{-e}{\hbar c}[\vec{r}(t) - \vec{r}(0)]B^2 + \frac{e}{\hbar c}\vec{B} \cdot [\vec{r}(t) - \vec{r}(0)]\vec{B}. \tag{L103}$$

de Haas–van Alphen Effect

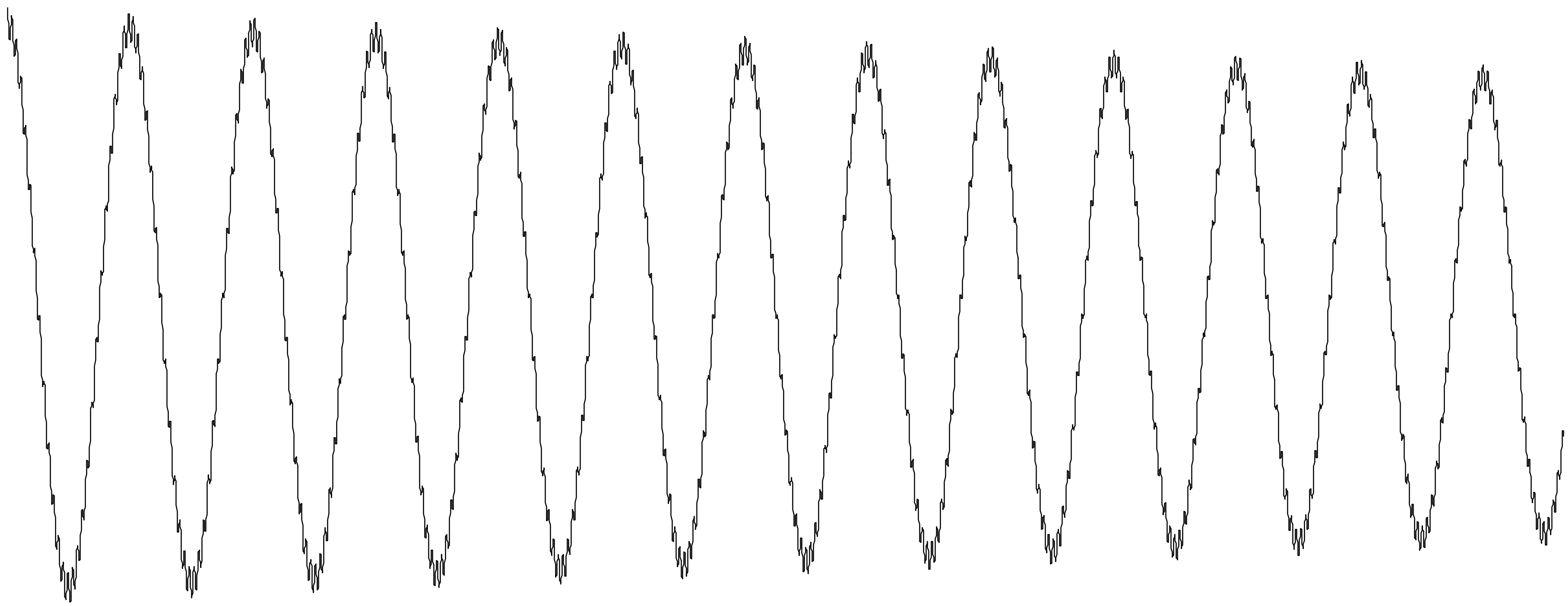


Figure 8: Sketch of de Haas–van Alphen oscillations of magnetization M in gold similar to those measured by Shoenberg and Vanderkooy (1970).

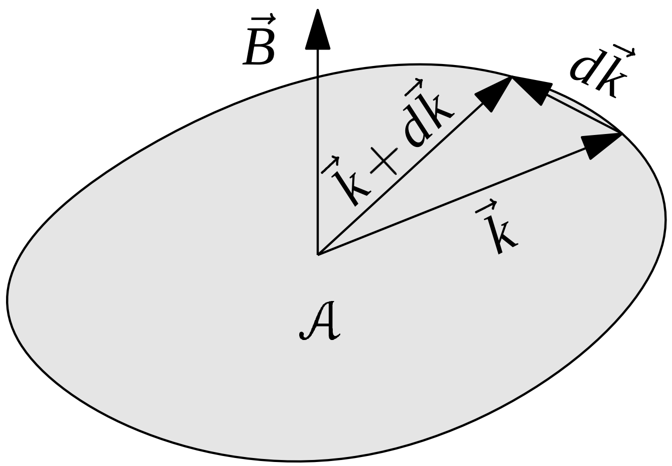
$$2\pi j \quad = \quad \Gamma - \int_0^{\mathcal{T}} dt \left[\frac{e\vec{A}}{c\hbar} \cdot \dot{\vec{r}} - \frac{e}{\hbar c} (\dot{\vec{r}} \times \vec{B}) \cdot \vec{r} \right] \tag{L104}$$

$$= \Gamma + \int_0^{\mathcal{T}} dt\, \frac{e}{2\hbar c} \vec{r} \cdot (\dot{\vec{r}} \times \vec{B}) \tag{L105}$$

$$= \Gamma + \int_0^{\mathcal{T}} dt\, \frac{\hbar c}{2eB} \left(\frac{\vec{B}}{B} \times \vec{k} \right) \cdot \dot{\vec{k}} \tag{L106}$$

$$= \Gamma + \oint d\vec{k} \cdot \frac{\hbar c}{2eB} \left(\frac{\vec{B}}{B} \times \vec{k} \right) \tag{L107}$$

$$\Rightarrow 2\pi j = \Gamma + \mathcal{A} \frac{\hbar c}{eB}, \tag{L108}$$



$$\frac{\mathcal{A}}{B} \frac{\hbar c}{2\pi e} = 1.05 \cdot 10^4 \frac{\mathcal{A} \cdot \text{Å}^2}{[B/\text{T}]} = j - \Gamma/2\pi \tag{L109a}$$

$$\Rightarrow \mathcal{A} \quad = \quad 9.52 \cdot 10^{-5} \frac{1}{\Delta(1/B)} [\text{Å}^{-2}/\text{T}]. \qquad \text{(L109b)}$$

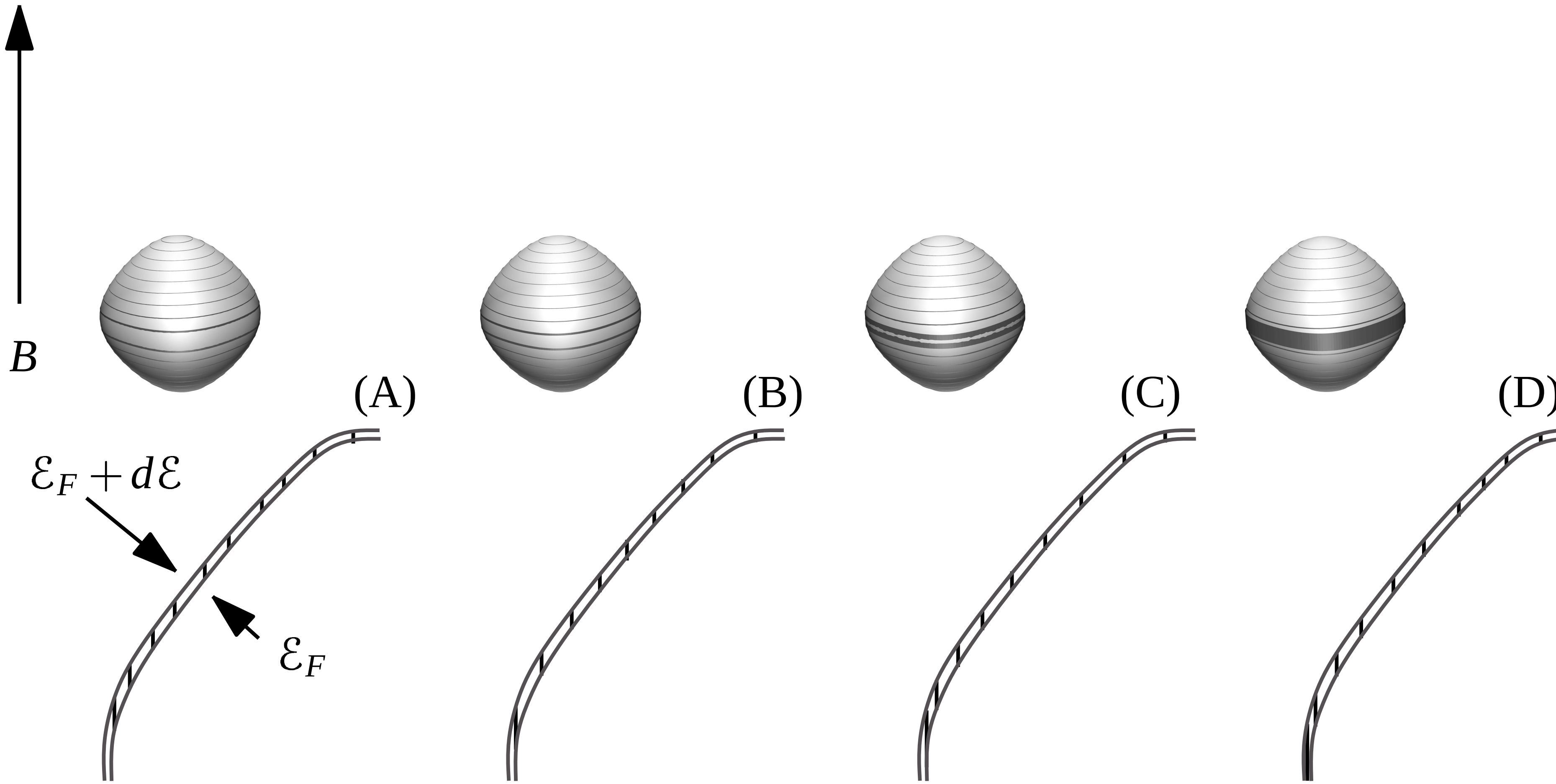


Figure 9:

Experimental Measurements of Fermi Surfaces

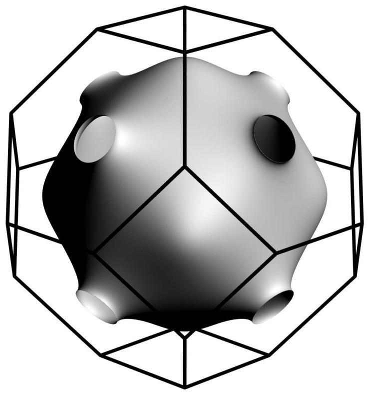


Figure 10: Fermi surface of copper, Shoenberg (1984).

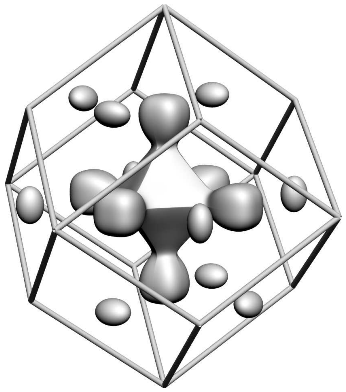


Figure 11: The Fermi surface of tungsten, Girvan et al. (1968).