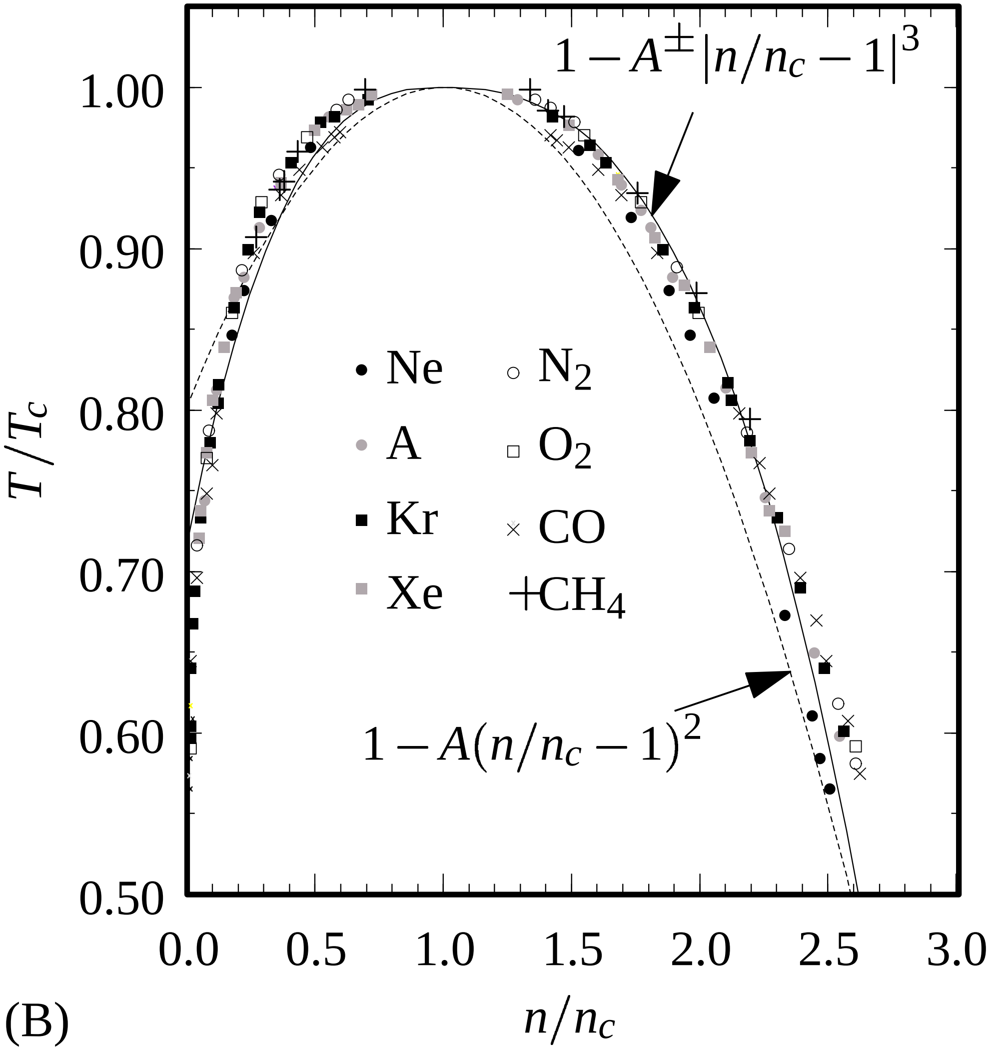
Classical Theories of Magnets



Definitions

- ☞ Phenomenology of Magnets
- ☞ Dipole Moments
- ☞ Ferromagnets, Ferrimagnets,and Antiferromagnets
- ☞ Mean Field Theory
- ☞ The Lenz–Ising Model
- ☞ Domains
- ☞ Hysteresis
- ☞ Order–Disorder Transitions
- ☞ Critical Phenomena
- ☞ Landau Free Energy
- ☞ Scaling and Universality

Magnetic Moments

$$\vec{j}_{\text{mag}} = c\vec{\nabla} \times \vec{M}. \tag{L1}$$

$$\vec{H} \equiv \vec{B} - 4\pi\vec{M} \tag{L2}$$

$$\begin{aligned}
\nabla \times \vec{B} &= \frac{4\pi\vec{j}_{\text{mag}}}{c} + \frac{4\pi\vec{j}_{\text{ext}}}{c} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} && \text{(L3)}\\
&= 4\pi\vec{\nabla} \times \vec{M} + \frac{4\pi\vec{j}_{\text{ext}}}{c} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} && \text{(L4)}\\
\Rightarrow \nabla \times \vec{H} &= \frac{4\pi\vec{j}_{\text{ext}}}{c} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}. && \text{(L5)}
\end{aligned}$$

$$\vec{B} = \mu\vec{H}, \tag{L6}$$

$$\chi = \frac{\partial M}{\partial H}. \tag{L7}$$

Conductivity

$$\vec{E}_{\rm L} = \frac{\vec{q}(\vec{E}\cdot\vec{q})}{q^2}, \quad \vec{E}_{\rm T} = \vec{E} - \vec{E}_{\rm L}. \tag{L8}$$

$$\vec{j} = \frac{c^2 q^2}{4\pi i\omega}\left(1 - \frac{1}{\mu}\right)\vec{E}_{\rm T} \tag{L9}$$

$$\Rightarrow \frac{\partial \vec{j}}{\partial t} = -\frac{c^2}{4\pi}\left(1 - \frac{1}{\mu}\right)\vec{\nabla}\times\vec{\nabla}\times\vec{E} \tag{L10}$$

$$= \frac{c}{4\pi} ? \qquad\qquad ? \tag{L11}$$

$$\Rightarrow \vec{j} = \frac{c}{4\pi} ? \qquad\qquad ? \tag{L12}$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = ? \qquad\qquad ? \tag{L13}$$

$$\Rightarrow \vec{\nabla}\times\frac{\vec{B}}{\mu} = \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}. \tag{L14}$$

Free Energy

$$\mathcal{E}\{\vec{B}(\vec{r})\} \tag{L15}$$

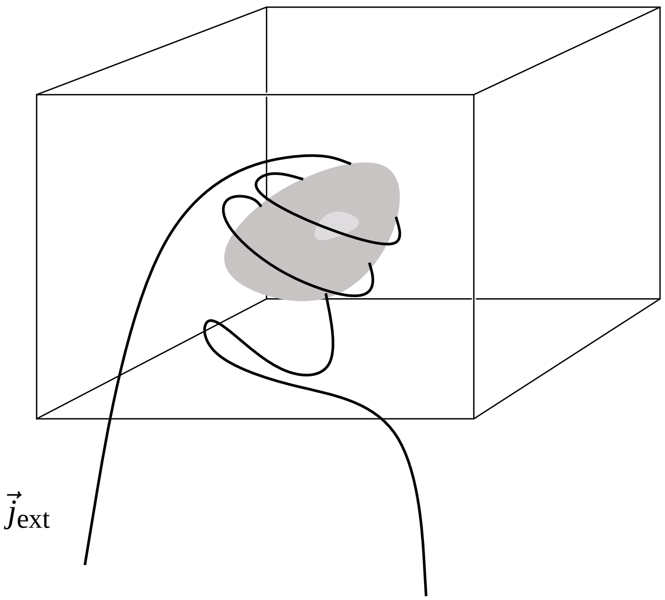


Figure 1: Sample influenced only by currents $\vec{j}_{\text{ext}}$.

$$\frac{d\mathcal{E}}{dt} = -\int d\vec{r}\,\vec{E}(\vec{r})\cdot\vec{j}_{\text{ext}}(\vec{r}). \tag{L16}$$

$$\vec{H}(\vec{r}) = 4\pi\frac{\delta\mathcal{E}\{\vec{B}\}}{\delta\vec{B}(\vec{r})}. \tag{L17}$$

Free Energy

$$\delta\mathcal{E} = \frac{1}{4\pi}\int d\vec{r}\,\vec{H}(\vec{r})\cdot\delta\vec{B}(\vec{r}). \tag{L18}$$

$$\frac{\partial\mathcal{E}}{\partial t} = \frac{1}{4\pi}\int d\vec{r}\,\vec{H}(\vec{r})\cdot\frac{\partial\vec{B}(\vec{r})}{\partial t} \tag{L19}$$

$$= -\frac{c}{4\pi}\int d\vec{r}\,\vec{H}\cdot\vec{\nabla}\times\vec{E} \tag{L20}$$

$$= -\frac{c}{4\pi}\int d\vec{r}\left[\vec{E}\cdot\vec{\nabla}\times\vec{H}-\vec{\nabla}\cdot(\vec{H}\times\vec{E})\right] \tag{L21}$$

$$= -\frac{c}{4\pi}\int d\vec{r}\,\vec{E}\cdot\vec{\nabla}\times\vec{H}. \tag{L22}$$

$$\vec{\nabla} \times \vec{H}(\vec{r}) = \frac{4\pi}{c}\vec{j}_{\rm ext}. \tag{L23}$$

$$\vec{M}(\vec{r}) \equiv \frac{1}{4\pi}\left(\vec{B}(\vec{r})-\vec{H}(\vec{r})\right); \tag{L24}$$

$$\mathcal{F}(T,\vec{B}) = \mathcal{E}(\vec{B}) - TS. \tag{L25}$$

$$\delta\mathcal{F} = -S\delta T + \int d\vec{r}\,\vec{H}(\vec{r})\cdot\delta\vec{M}(\vec{r}) + \frac{1}{8\pi}\int d\vec{r}\,\delta H^2(\vec{r}). \tag{L26}$$

$$\tilde{\mathcal{G}} = \mathcal{F} - \frac{1}{4\pi}\int d\vec{r}\,\vec{B}(\vec{r})\cdot\vec{H}(\vec{r}). \tag{L27}$$

$$\delta\tilde{\mathcal{G}} = -\frac{1}{4\pi}\int d\vec{r}\,\vec{B}(\vec{r})\cdot\delta\vec{H}(\vec{r}) \tag{L28}$$

$$= -\int d\vec{r}\,\vec{M}(\vec{r})\cdot\delta\vec{H}(\vec{r}) - \frac{1}{4\pi}\int d\vec{r}\,\vec{H}(\vec{r})\cdot\delta\vec{H}(\vec{r}). \tag{L29}$$

$$\mathcal{G} = \tilde{\mathcal{G}} + \frac{1}{8\pi}\int d\vec{r}\,H^2(\vec{r}) \tag{L30}$$

$$\delta\mathcal{G} = -S\delta T - \int d\vec{r}\,\vec{M}\cdot\delta\vec{H}. \tag{L31}$$

Magnetic Dipole Moments

Element	χ $(10^{-6}\,\text{cm}^3\,\text{mole}^{-1})$	Element	χ $(10^{-6}\,\text{cm}^3\,\text{mole}^{-1})$
Ar	−19.18	N2	−12.04
As	−5.24	Ne	−7.02
B	−6.70	P	−26.63
C	−5.88	S	−15.39
Cl	−20.18	Se	−23.69
Ge	−7.99	Si	−3.09
H2	−4.00	Te	−37.00
He	−1.88	Tl	−43.42
I	−45.68	Xe	−43.33
Kr	−28.49		

Magnetic Dipole Moments

$$\vec{m} = \int d\vec{r}\,\frac{1}{2c}\vec{r}\times\vec{j}(\vec{r}). \tag{L32}$$

$$\vec{F} = \frac{1}{c}\int d\vec{r}\,\vec{j}(\vec{r})\times\vec{B}(\vec{r}). \tag{L33}$$

$$\begin{aligned}
\vec{F} &= \frac{1}{c}\int d\vec{r}\,\vec{j}(\vec{r})\times[\vec{B}(0)+(\vec{r}\cdot\vec{\nabla})\vec{B}(0)+\ldots] && \text{(L34)}\\
&= 0+(\vec{m}\times\vec{\nabla})\times\vec{B} && \text{(L35)}\\
&= \vec{\nabla}(\vec{m}\cdot\vec{B}) && \text{(L36)}\\
\Rightarrow U &= -\vec{m}\cdot\vec{B}. && \text{(L37)}
\end{aligned}$$

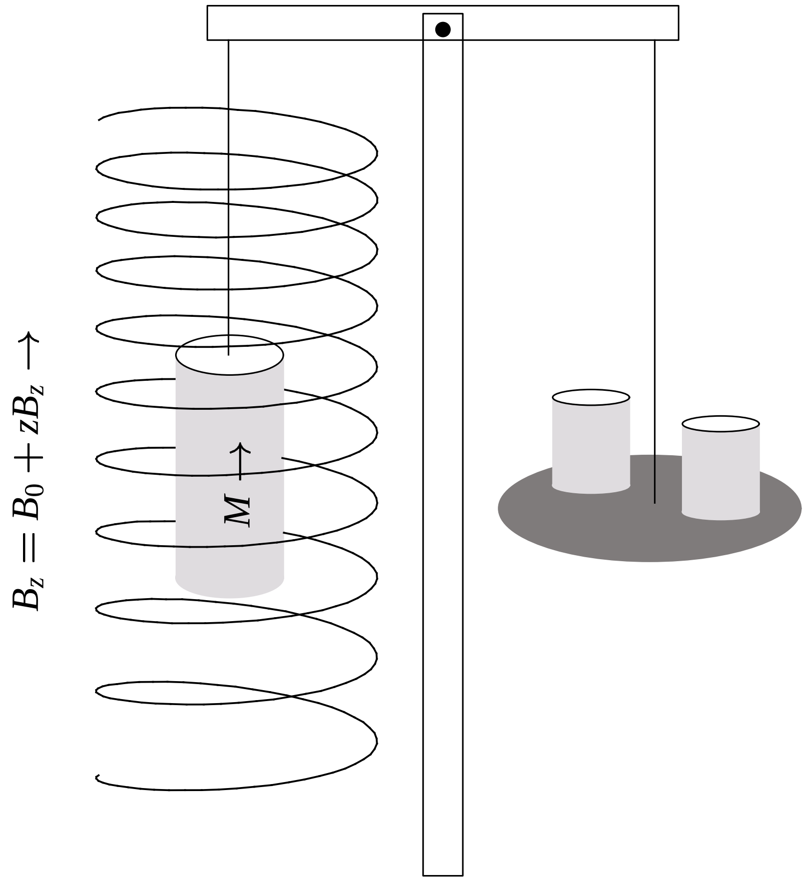


Figure 2: Schematic view of Faraday balance.

$$B_z(z) = B_0 + zB_1. \tag{L38}$$

Spontaneous Magnetization of Ferromagnets

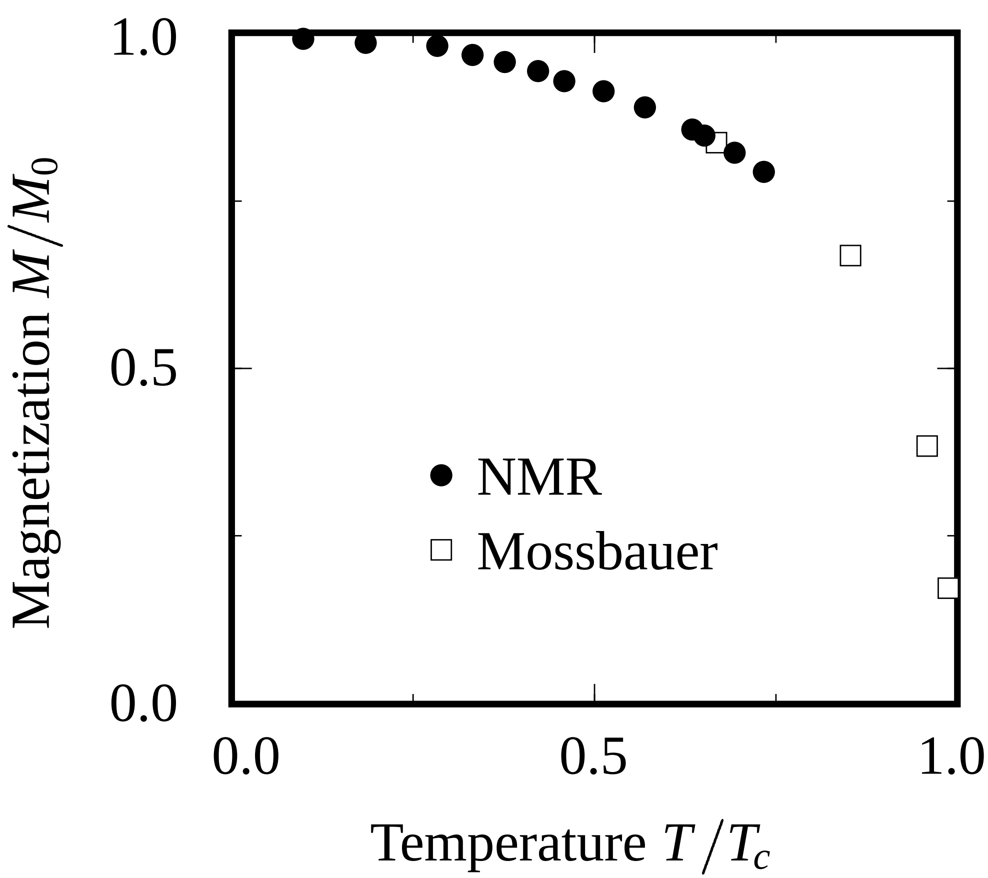


Figure 3: Internal magnetic fields in iron [Source:Preston et al. (1962).]

$$\mu_B = e\hbar/2mc, \tag{L39}$$

$$\mu_B = 9.27\cdot 10^{-21}\text{cm esu} \;=\; 9.27\cdot 10^{-21}\text{erg G}^{-1}. \tag{L40}$$

$$\chi \propto \frac{1}{T-\Theta}; \tag{L41}$$

Spontaneous Magnetization of Ferromagnets

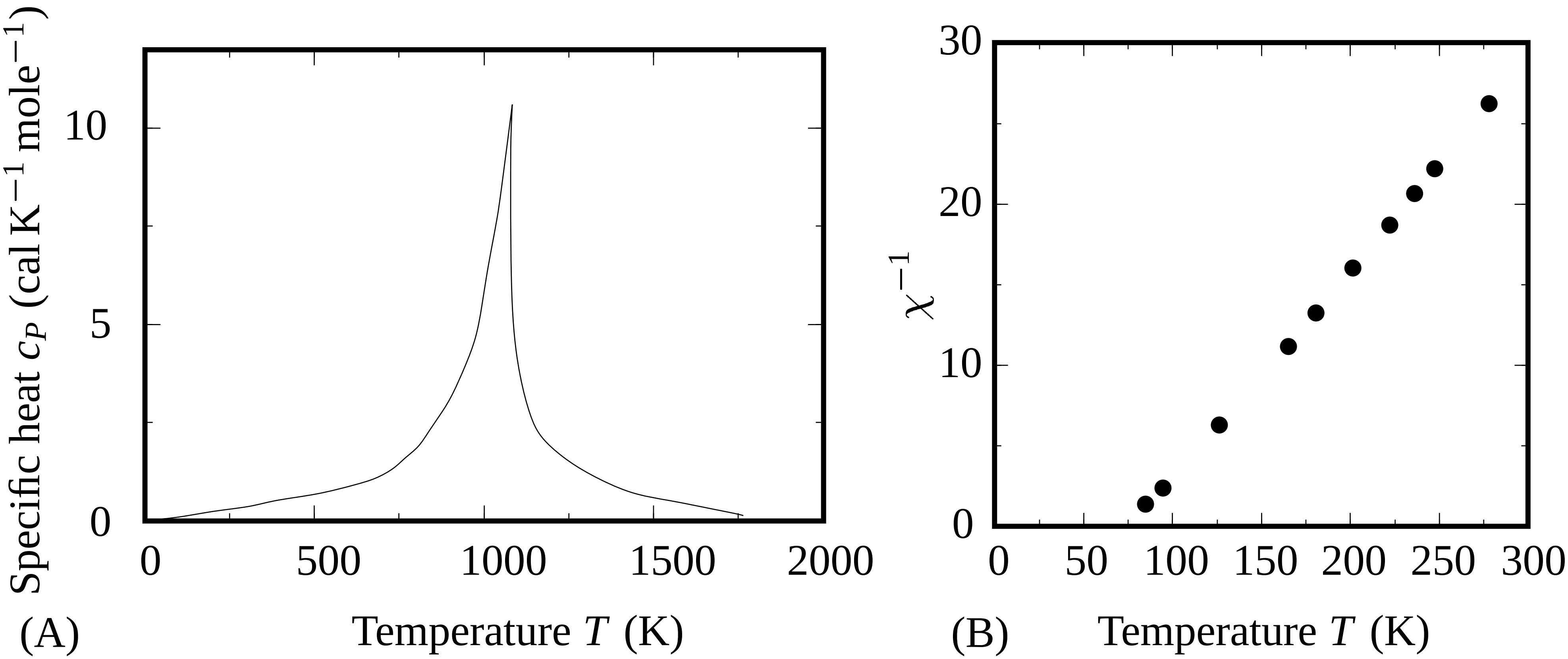


Figure 4: (A) Specific heat of iron. [Source: Hofmann et al. (1956) p. 53.] (B) Magnetic susceptibility χ of EuO. Source: Matthias et al. (1961), p. 160.]

Spontaneous Magnetization of Ferromagnets

Compound		T_c (K)	Θ (K)	m_I (μ_B)
Cr	a	312		0.59
CoO	a	291	−330	3.8
CuO	a	230	−745	0.5
Mn	a	100		0.5
MnO	a	122	−610	5
NiO	a	523	−2470	2
O_2	a	23.9		2
Co	f	1394	1415	1.72
Dy	f	85	157	10.65
Eu	f	289	108	7.12
Fe	f	1043	1100	2.2
Gd	f	302	289	7.97
Ho	f	20	87	10.9
Ni	f	628	650	0.6
Tb	f	20	87	10.9

Compound		T_c (K)	m_I (μ_B)
$FeFe_2O_4$ (magnetite)	fi	858	4.1
$FeNiFeO_4$	fi	858	2.3
$FeLiFeO_4$	fi	943	2.6
$FeCuFeO_4$	fi	728	1.3
$FeCoFeO_4$	fi	793	3.7

Ferrimagnets

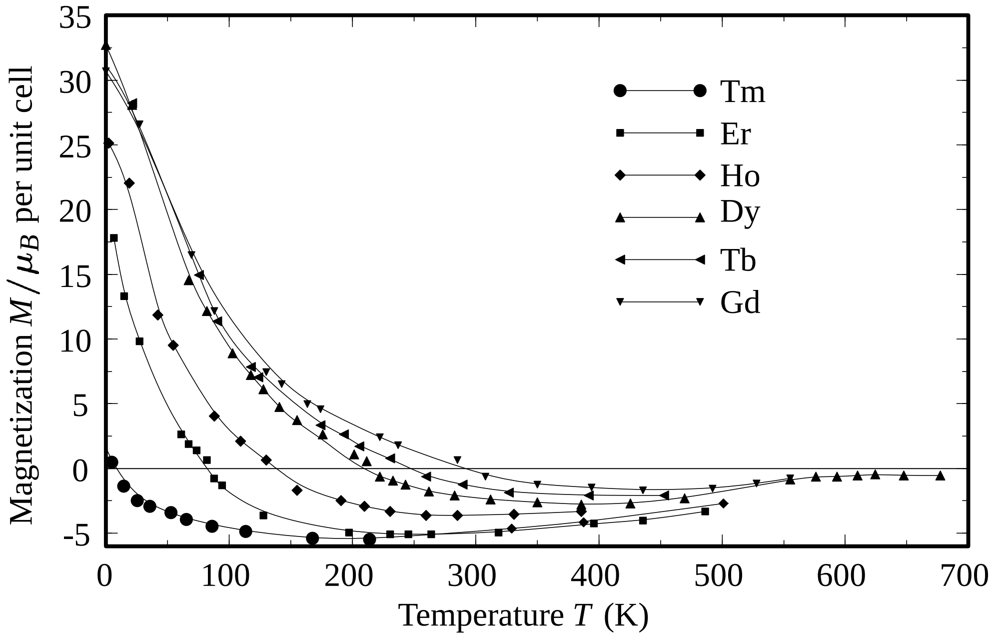


Figure 5: Spontaneous magnetization of rare earth iron garnets $5Fe_2O_3 \cdot R_2O_2$ [Source: Bertaut and Pauthenet (1957).]

$$\chi = \frac{1}{T + |\Theta|}. \tag{L42}$$

Antiferromagnets

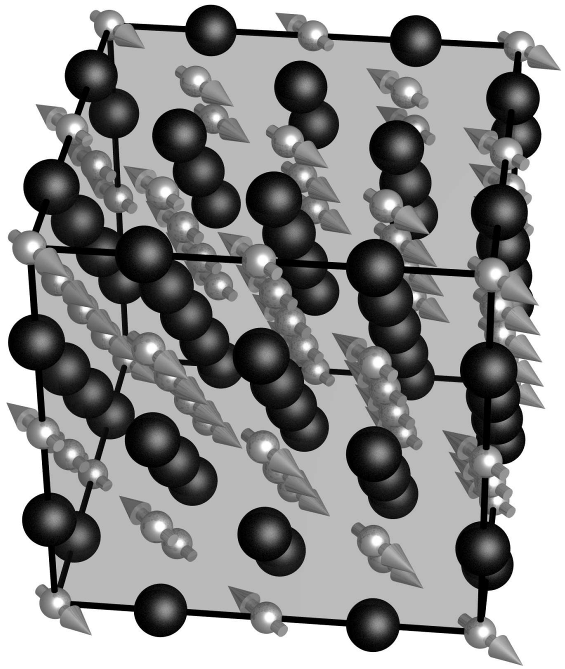


Figure 6: Spin structure of transition metal oxides such as CoO or NiO.

$$\mathcal{E} = -\sum_{\langle \vec{R}\vec{R}' \rangle} J\sigma_{\vec{R}}\sigma_{\vec{R}'} - \sum_{\vec{R}} H\mu_B\sigma_{\vec{R}}, \tag{L43}$$

$$\mathcal{P}(\sigma_{\vec{R}}) \propto \exp\left\{\beta \sum_{\langle \vec{R}\vec{R}' \rangle} J\sigma_{\vec{R}}\sigma_{\vec{R}'} + \beta \sum_{\vec{R}} H\mu_B\sigma_{\vec{R}}\right\}. \tag{L44}$$

$$\sigma_{\vec{R}} = \bar{\sigma} + (\sigma_{\vec{R}} - \bar{\sigma}), \tag{L45}$$

$$\sigma_{\vec{R}}\sigma_{\vec{R}'} = [\bar{\sigma} + (\sigma_{\vec{R}} - \bar{\sigma})][\bar{\sigma} + (\sigma_{\vec{R}'} - \bar{\sigma})] \approx \bar{\sigma}(\sigma_{\vec{R}} + \sigma_{\vec{R}'}) - \bar{\sigma}^2. \tag{L46}$$

$$-\sum_{\langle \vec{R}\vec{R}' \rangle} J\sigma_{\vec{R}}\sigma_{\vec{R}'} - \sum_{\vec{R}} H\mu_B\sigma_{\vec{R}} \approx NzJ\bar{\sigma}^2/2 - \sum_{\vec{R}} (H + \bar{H})\mu_B\sigma_{\vec{R}} \tag{L47}$$

$$\bar{H} = \frac{zJ\bar{\sigma}}{\mu_B}. \tag{L48}$$

$$\begin{aligned}
Z &\approx \sum_{\sigma_1\ldots\sigma_N} \exp\left[-\beta(NzJ\bar{\sigma}^2/2 - \sum_{\vec{R}}(H+\bar{H})\mu_B\sigma_{\vec{R}})\right] && \text{(L49)}\\
&= e^{-\beta NzJ\bar{\sigma}^2/2}\left[\exp[\beta(H+\bar{H})\mu_B] + \exp[-\beta(H+\bar{H})\mu_B]\right]^N && \text{(L50)}\\
\Rightarrow \mathcal{F} &= -k_BT\ln Z = NzJ\bar{\sigma}^2/2 - Nk_BT\ln[2\cosh\beta\mu_B(H+\bar{H})]. && \text{(L51)}
\end{aligned}$$

$$\begin{aligned}
\bar{\sigma} &= \frac{1}{Z}\sum_{\sigma_1\ldots\sigma_N}\frac{1}{N}\sum_{\vec{R}'}\sigma_{\vec{R}'}\exp[-\beta\mathcal{E}\{\sigma_{\vec{R}}\}] && \text{(L52)}\\
&= \frac{1}{Z}\sum_{\sigma_1\ldots\sigma_N}\frac{1}{\beta N\mu_B}\frac{\partial}{\partial H}\exp[-\beta\mathcal{E}\{\sigma_{\vec{R}}\}] && \text{(L53)}\\
&= -\frac{1}{N}\frac{1}{\mu_B}\frac{\partial\mathcal{F}}{\partial H} && \text{(L54)}\\
&= \;? \qquad\qquad\qquad\qquad ? && \text{(L55)}
\end{aligned}$$

$$\Rightarrow \bar{\sigma} = \tanh \beta[zJ\bar{\sigma} + \mu_B H]. \tag{L56}$$

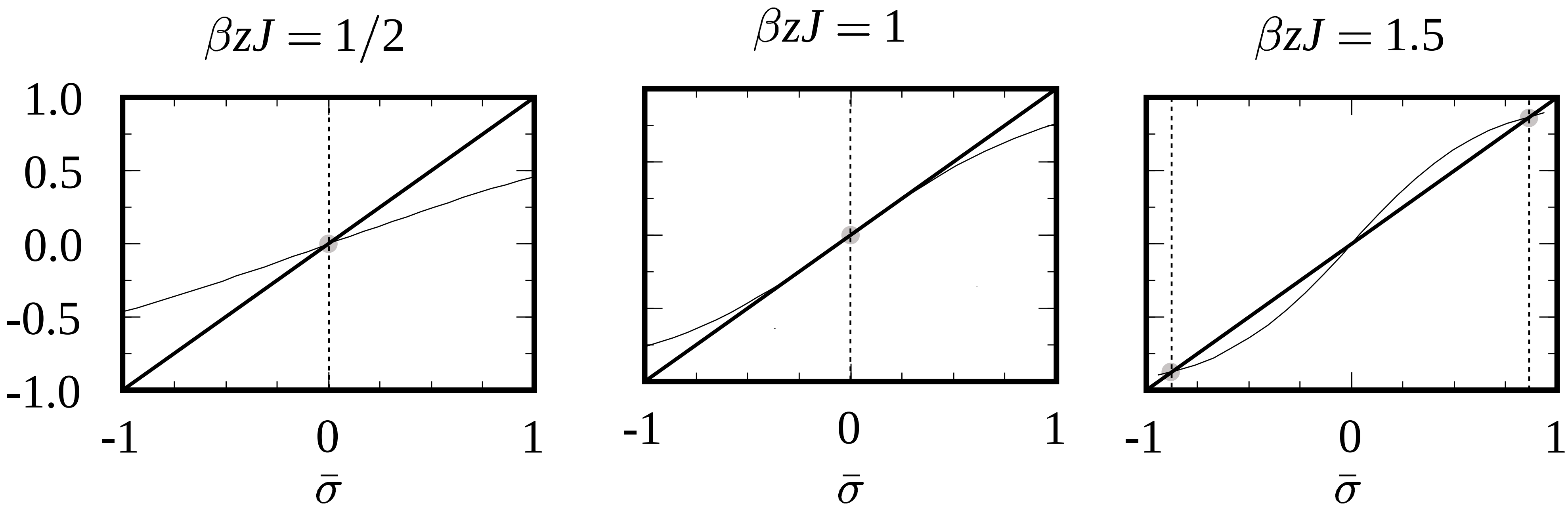


Figure 7: Graphical solution of Eq. (L56).

Domains

$$\mathcal{E} = -\sum_{\langle \vec{R}\vec{R}' \rangle} J\vec{\sigma}_{\vec{R}} \cdot \vec{\sigma}_{\vec{R}'} + \sum_{\vec{R}} \left[\alpha(\vec{\sigma}_{\vec{R}} \cdot \hat{x})^2 - \mu_B \vec{B} \cdot \vec{\sigma}_{\vec{R}} \right] + \frac{1}{8\pi} \int d\vec{r}\, \vec{B} \cdot \vec{B}. \tag{L57}$$

$$\mathcal{E} = \frac{JL}{la} + \frac{\alpha l}{a^2} \tag{L58}$$

$$\Rightarrow \frac{\partial \mathcal{E}}{\partial l} = \frac{\alpha}{a^2} - \frac{JL}{l^2 a} \tag{L59}$$

$$\Rightarrow l = a\sqrt{\frac{JL}{\alpha a}} \tag{L60}$$

$$\Rightarrow \frac{\mathcal{E}_{\text{min}}}{L} = 2\frac{\sqrt{\alpha J}}{a^2}\sqrt{\frac{a}{L}}. \tag{L61}$$

Domains

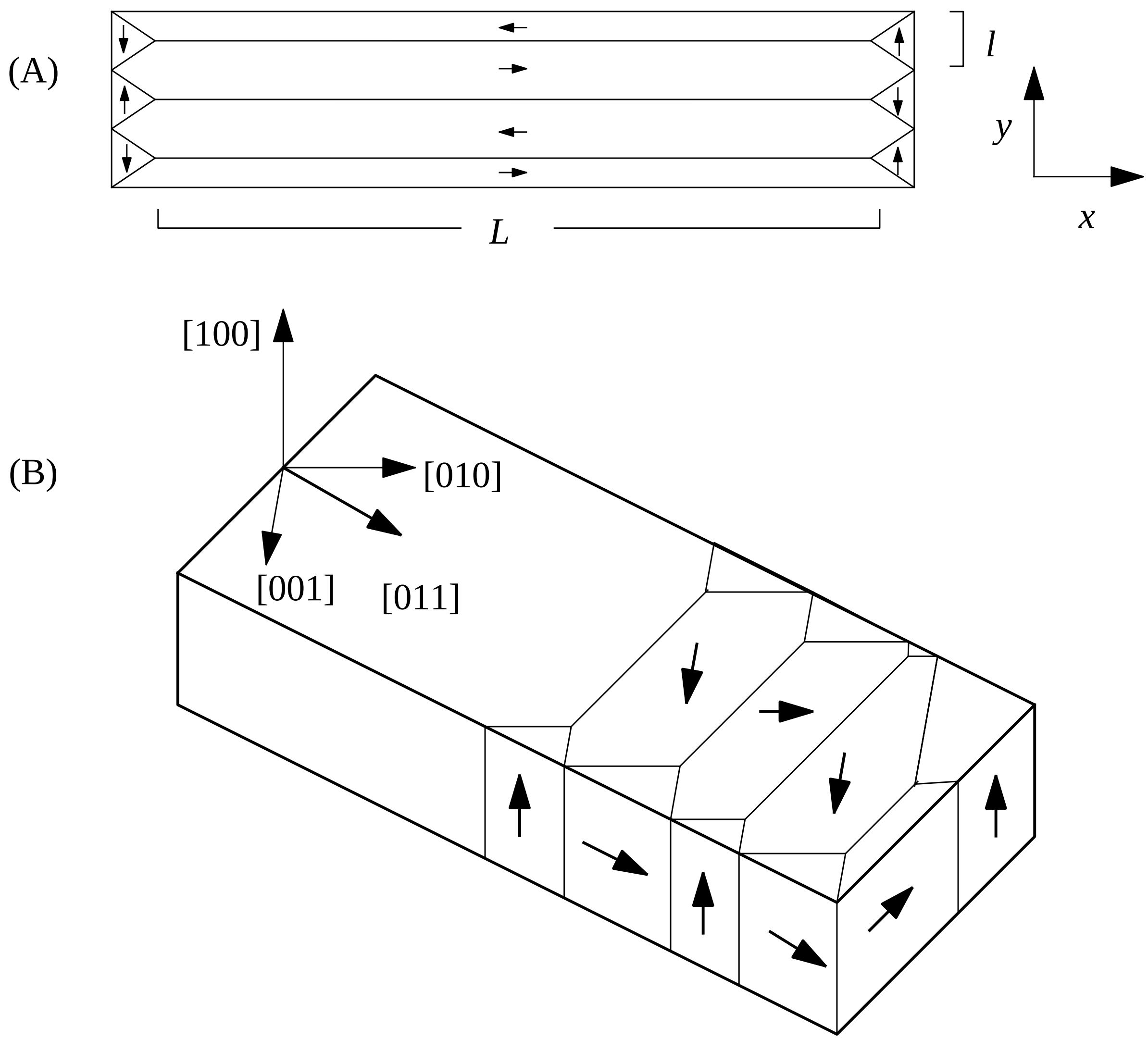


Figure 8: (A) Domain formation in a rectangular bar magnet. (B) In an anisotropic crystal

Hysteresis

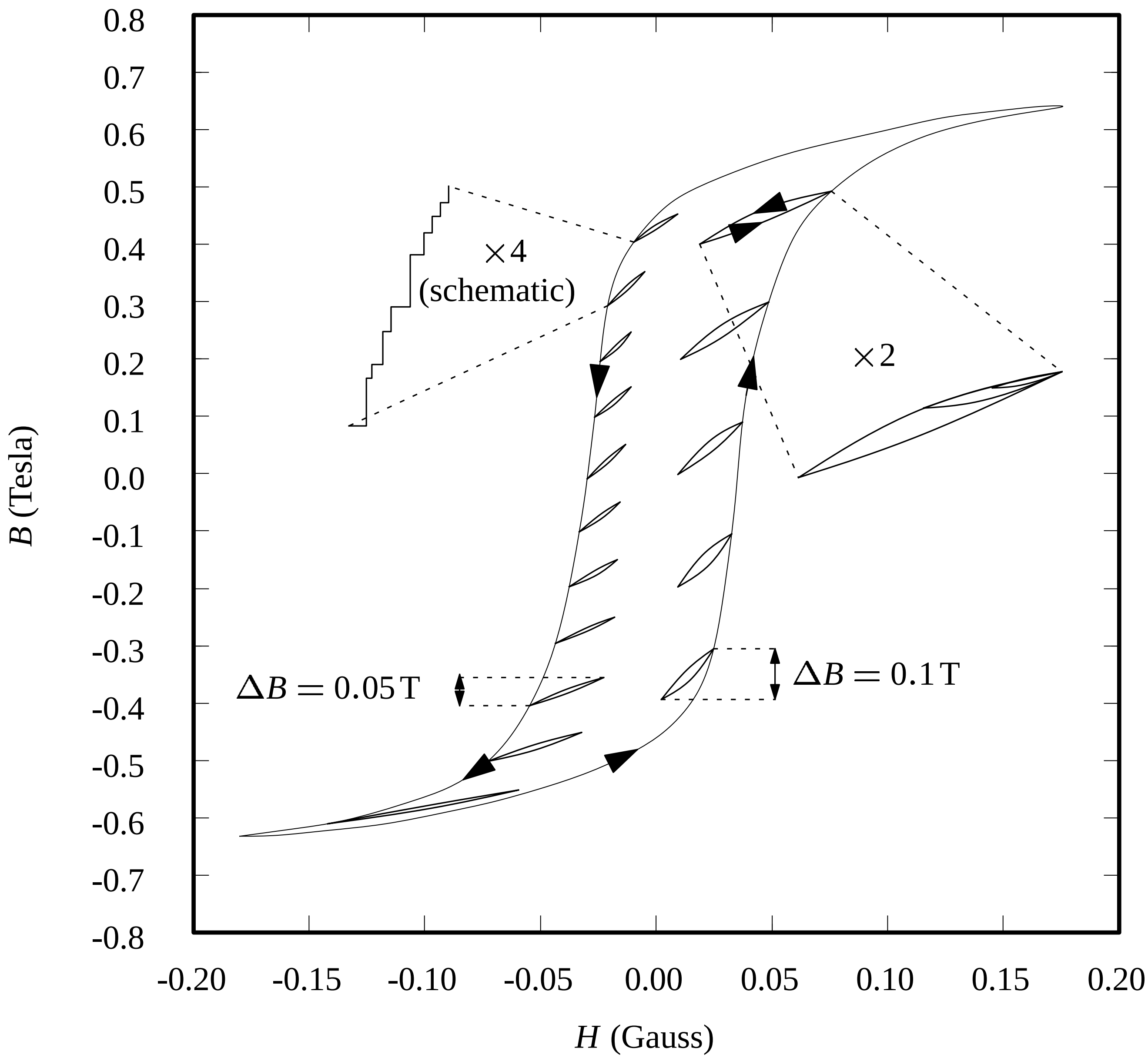


Figure 9: Hysteresis in the magnetization curve of Permalloy. [Source: Bozorth (1951)]

$$f(-1,-1)=\epsilon_{AA},\ f(1,-1)=f(-1,1)=\epsilon_{AB},\quad \text{and}\quad f(1,1)=\epsilon_{BB}. \tag{L62}$$

$$f(\sigma_{\vec{R}},\sigma_{\vec{R}'})=C_1+C_2(\sigma_{\vec{R}}+\sigma_{\vec{R}'})+C_3\sigma_{\vec{R}}\sigma_{\vec{R}'}, \tag{L63}$$

$$f(\sigma_{\vec{R}},\sigma_{\vec{R}'}) = \frac{\epsilon_{BB}+\epsilon_{AB}}{2}(\sigma_{\vec{R}}+\sigma_{\vec{R}'})-\epsilon_{AB}\sigma_{\vec{R}}\sigma_{\vec{R}'}. \tag{L64}$$

$$\mathcal{P}(\sigma_{\vec{R}}) = \exp\left\{\beta\mu\sum_{\vec{R}}\sigma_{\vec{R}}-\beta\sum_{\langle\vec{R}\vec{R}'\rangle}f(\sigma_{\vec{R}},\sigma'_{\vec{R}})\right\} \tag{L65}$$

$$= \exp\left\{\beta\mu_B H\sum_{\vec{R}}\sigma_{\vec{R}}+\beta J\sum_{\langle\vec{R}\vec{R}'\rangle}\sigma_{\vec{R}}\sigma_{\vec{R}'}\right\} \tag{L66}$$

where

$$\mu_B H = \mu - \frac{\epsilon_{BB} + \epsilon_{AB}}{2} z \quad \text{and} \quad J = \epsilon_{AB}. \tag{L67}$$

$$\sigma_{\vec{R}_A} = \sigma_A + (\sigma_{\vec{R}_A} - \sigma_A), \sigma_{\vec{R}_B} = \sigma_B + (\sigma_{\vec{R}_B} - \sigma_B) \tag{L68}$$

$$\mathcal{P} = \exp\left\{\beta\mu_B H \sum_{\vec{R}} \sigma_{\vec{R}} + \beta J \sum_{\langle \vec{R}_A \vec{R}_B \rangle} (\sigma_A \sigma_{\vec{R}_B} + \sigma_B \sigma_{\vec{R}_A} - \sigma_A \sigma_B)\right\}, \tag{L69}$$

$$= \prod_{\vec{R}_A} \exp\left\{\beta\mu_B H \sigma_{\vec{R}_A} + \beta J z(\sigma_B \sigma_{\vec{R}_A} - \sigma_A \sigma_B/2)\right\}$$
$$\prod_{\vec{R}_B} \exp\left\{\beta\mu_B H \sigma_{\vec{R}_B} + \beta J z(\sigma_A \sigma_{\vec{R}_B} - \sigma_A \sigma_B/2)\right\}. \tag{L70}$$

$$\sigma_A = \left\langle \sigma_{\vec{R}_A} \right\rangle = \frac{e^{\{\beta\mu_B H + \beta J z \sigma_B\}} - e^{\{-\beta\mu_B H - \beta J z \sigma_B\}}}{e^{\{\beta\mu_B H + \beta J z \sigma_B\}} + e^{\{-\beta\mu_B H - \beta J z \sigma_B\}}}. \quad \text{(L71)}$$

$$\sigma_A = \tanh[\beta\mu_B H + \beta z \sigma_B J] \quad \text{(L72a)}$$

$$\sigma_B = \tanh[\beta\mu_B H + \beta z \sigma_A J]. \quad \text{(L72b)}$$

$$\sigma_A + \sigma_B = 0. \quad \text{(L73)}$$

$$\sigma_A = -\tanh(\beta J z \sigma_A) = \tanh(\beta |J| z \sigma_A). \quad \text{(L74)}$$

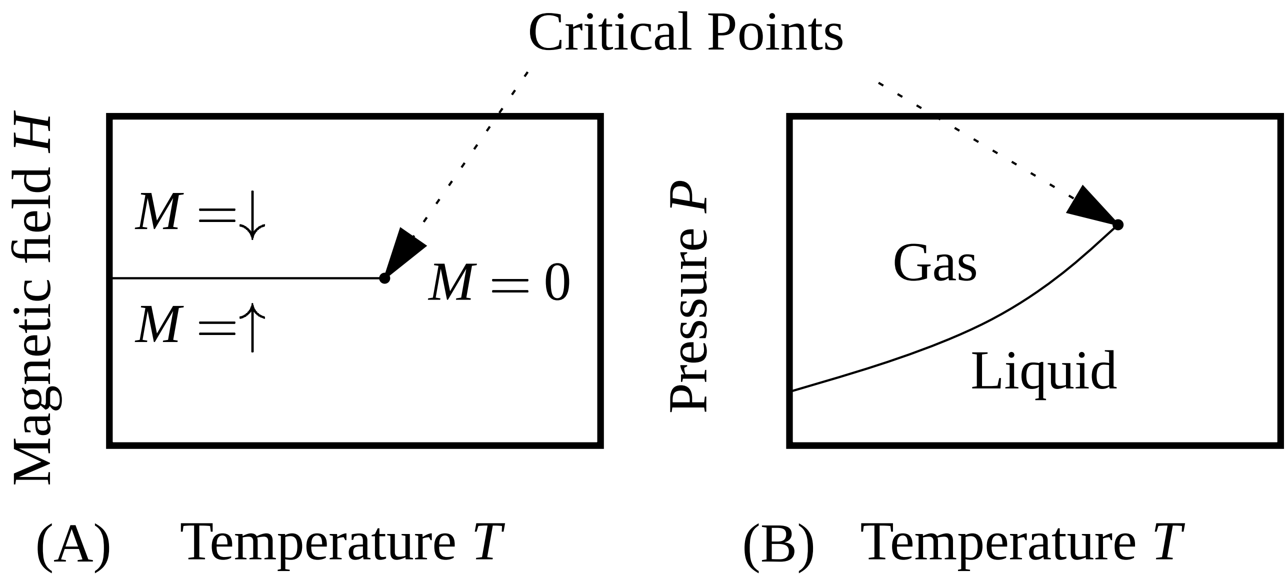


Figure 10: (A) Schematic phase diagram for a ferromagnet. (B) Schematic phase diagram of liquid–gas system. .

Landau Free Energy

$$\mathcal{F}(M,T) = A_0(T) + A_2(T)M^2 + A_4(T)M^4 + HM. \qquad \text{(L75)}$$

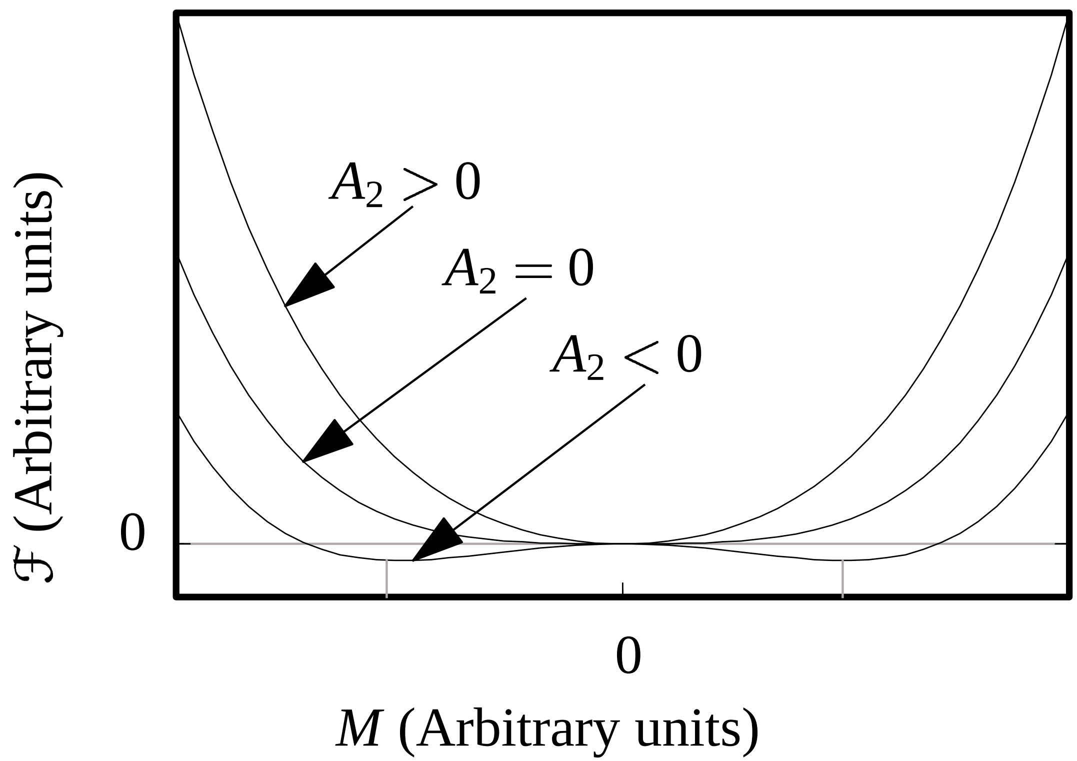


Figure 11: Landau free energy, Eq. (L75), for $A_2 > 0$, $A_2 = 0$, and $A_2 < 0$.

$$t \equiv \frac{T - T_c}{T_c}, \qquad \text{(L76)}$$

$$\mathcal{F} = a_2 t M^2 + a_4 M^4 + HM. \tag{L77}$$

$$H + 2ta_2 M + 4a_4 M^3 = 0. \tag{L78}$$

$$M = \begin{cases} ? \qquad ? & \text{for } t < 0 \\ 0 & \text{for } t > 0. \end{cases} \tag{L79}$$

$$C_\mathcal{V} = \frac{\partial \mathcal{E}}{\partial T} = \frac{\partial}{\partial T}\frac{\partial \beta \mathcal{F}}{\partial \beta} \tag{L80}$$

$$= -\frac{1}{T_c}\frac{\partial}{\partial t}(1+t)^2 \frac{\partial}{\partial t}\left(\frac{\mathcal{F}}{1+t}\right) \tag{L81}$$

$$\approx -\frac{1}{T_c}\frac{\partial^2 \mathcal{F}}{\partial t^2} \tag{L82}$$

$$= \begin{cases} ? \qquad ? & \text{for } t < 0 \\ 0 & \text{for } t > 0. \end{cases} \tag{L83}$$

$$M = \sqrt{\frac{2|t|a_2}{4a_4}} + qH, \tag{L84}$$

$$q = -\frac{1}{4a_2|t|}. \tag{L85}$$

$$\frac{\partial M}{\partial H} \approx \begin{cases} -\dfrac{1}{4|t|a_2} & \text{for } t < 0 \\ -\dfrac{1}{2ta_2} & \text{for } t > 0. \end{cases} \tag{L86}$$

$$H + 4a_4M^3 = 0 \Rightarrow M \propto H^{1/3}. \tag{L87}$$

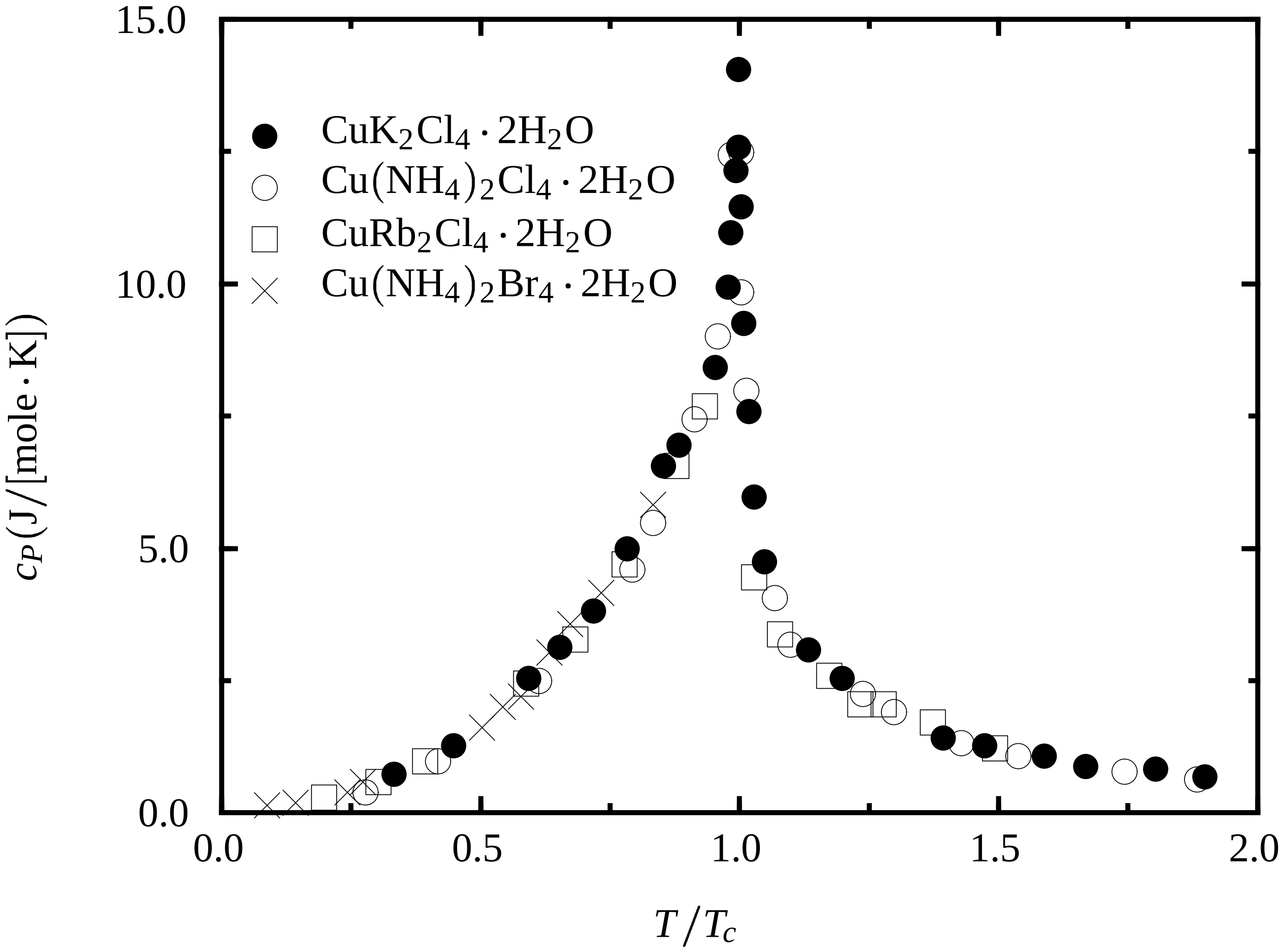


Figure 12: Molar heat capacities of four ferromagnetic copper salts versus scaled temperature T/T_c. [Source Jongh and Miedema (1974).]

Critical Phenomena

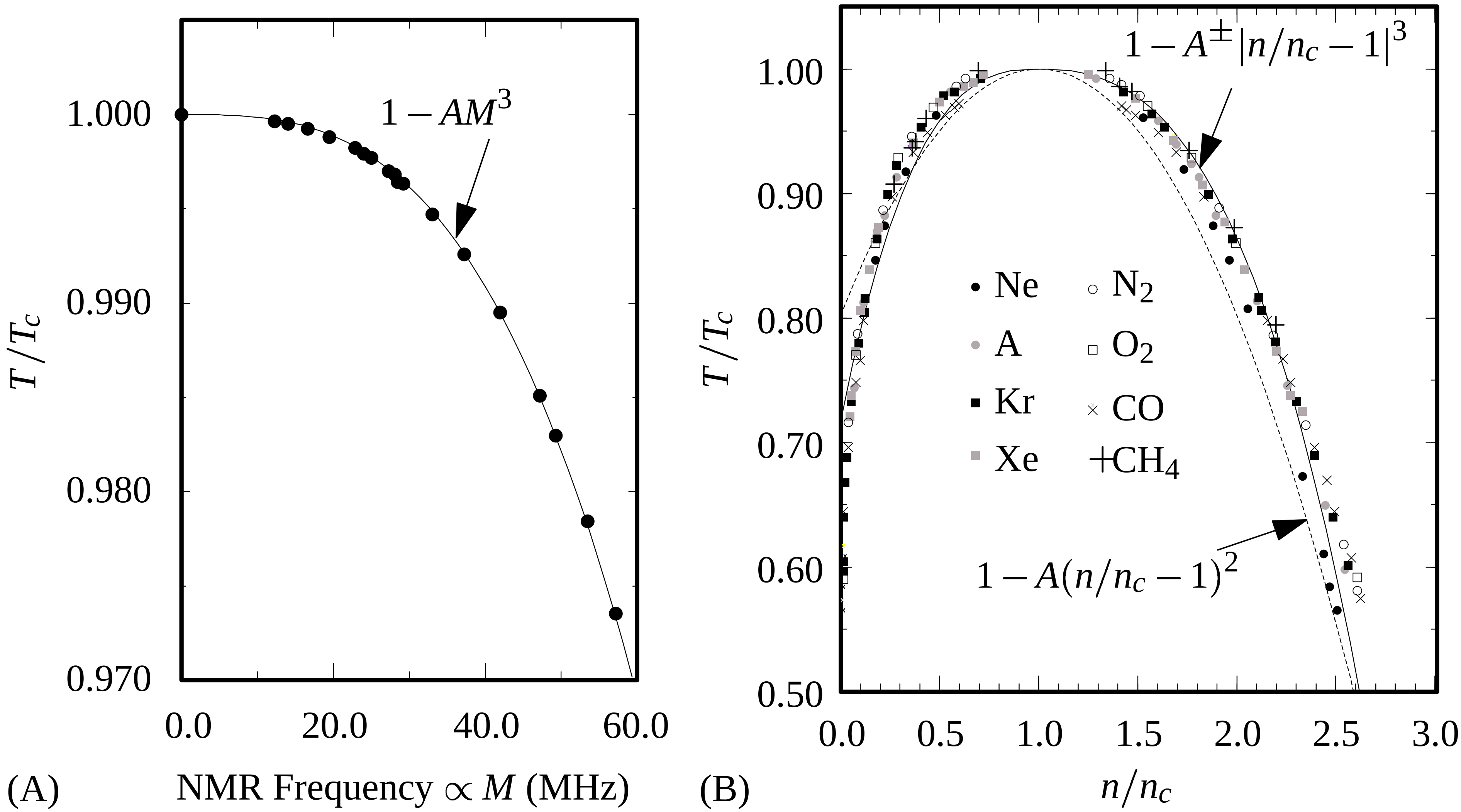


Figure 13: (A) Temperature versus magnetization, antiferromagnet Source: Heller and Benedek (1962) (B) Coexistence curve for eight fluids. Source: Guggenheim (1945).

$$dP = sdT + nd\mu, \tag{L88}$$

$$C_V(t) \sim |t|^{-\alpha}; \tag{L89}$$

$$M \sim |t|^{\beta} \quad \text{and} \quad \Delta n \sim |t|^{\beta}. \tag{L90}$$

$$K_T = \frac{1}{n}\frac{\partial n}{\partial P} \sim \frac{1}{n_c}\frac{\partial \Delta n}{\partial P} \sim |t|^{-\gamma}. \tag{L91}$$

$$\frac{\partial M}{\partial H} = \chi \sim |t|^{-\gamma}. \tag{L92}$$

$$P \sim |\Delta n|^{\delta}, \tag{L93}$$

$$|M| \sim |H|^{1/\delta}. \tag{L94}$$

$$g(r) - 1 \sim e^{-r/\xi} \tag{L95}$$

$$S(\vec{q}) - 1 = n \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}} [g(r) - 1] \tag{L96}$$

$$\sim \int d\vec{r}\, e^{-r/\xi + i\vec{q}\cdot\vec{r}} \sim \frac{1}{1 + \xi^2 q^2}. \tag{L97}$$

$$\xi \sim |t|^{-\nu}. \tag{L98}$$

$$g(r) \sim r^{-1-\eta}, \tag{L99}$$

Critical Phenomena

Exponent	Fluid	Magnet	Mean Field Theory	Experiment	3d Ising
α	$C_V \sim \lvert t\rvert^{-\alpha}$	$C_V \sim \lvert t\rvert^{-\alpha}$	discontinuity	0.11–0.12	0.110
β	$\Delta n \sim \lvert t\rvert^{\beta}$	$M \sim \lvert t\rvert^{\beta}$	$\frac{1}{2}$	0.35–0.37	0.325
γ	$K_T \sim \lvert t\rvert^{-\gamma}$	$\chi \sim \lvert t\rvert^{-\gamma}$	1	1.21–1.35	1.241
δ	$P \sim \lvert\Delta n\rvert^{\delta}$	$\lvert H\rvert \sim \lvert M\rvert^{\delta}$	3	4.0–4.6	4.82
ν	$\xi \sim \lvert t\rvert^{-\nu}$	$\xi \sim \lvert t\rvert^{-\nu}$		0.61–0.64	0.63
η	$g(r) \sim r^{-1-\eta}$	$g(r) \sim r^{-1-\eta}$		0.02–0.06	0.032

Scaling Theory

$$\frac{\mathcal{G}}{\mathcal{V}k_BT} = |t|^{x_1}G(t,H), \tag{L100}$$

$$C_{\mathcal{V}} = \frac{\partial}{\partial T}\frac{\partial\beta\mathcal{G}}{\partial\beta} \quad \sim \quad t^{-\alpha} \tag{L101}$$

$$\Rightarrow x_1 \quad = \quad 2-\alpha. \tag{L102}$$

$$G(t,H) = G\left(\frac{H}{H_0|t|^{\Delta}}\right). \tag{L103}$$

$$\lim_{y\to\infty} G(y) \sim y^{x_2}. \tag{L104}$$

$$\frac{\mathcal{G}}{\mathcal{V}k_BT} \sim |t|^{2-\alpha}\left(\frac{H}{H_0|t|^{\Delta}}\right)^{x_2} \sim |t|^{2-\alpha-\Delta x_2}. \tag{L105}$$

$$x_2 = \frac{2-\alpha}{\Delta}. \tag{L106}$$

$$-M = \frac{\partial \mathcal{G}}{\partial H} = |t|^{2-\alpha} \frac{1}{H_0|t|^{\Delta}} G'\left(\frac{H}{H_0|t|^{\Delta}}\right). \quad \text{(L107)}$$

$$|t|^{2-\alpha-\Delta} \sim |t|^{\beta} \quad \text{(L108)}$$

$$\Rightarrow \Delta = 2-\alpha-\beta. \quad \text{(L109)}$$

$$\left.\frac{\partial M}{\partial H}\right|_{H=0} = \chi \sim \frac{|t|^{2-\alpha}}{H_0^2|t|^{2\Delta}} G''(\frac{H}{H_0|t|^{\Delta}})\Big|_{H=0} \quad \text{(L110)}$$

$$\Rightarrow |t|^{2-\alpha-2\Delta} \sim |t|^{-\gamma} \quad \text{(L111)}$$

$$\Rightarrow \gamma = \alpha+2\Delta-2. \quad \text{(L112)}$$

$$2 = \alpha+2\beta+\gamma. \quad \text{(L113)}$$

$$M \sim \frac{1}{H_0|t|^\Delta}|t|^{2-\alpha}\left(\frac{H}{H_0|t|^\Delta}\right)^{x_2-1} \quad \text{(L114)}$$

$$\sim H^{x_2-1} = H^{(2-\alpha-\Delta)/\Delta} \quad \text{(L115)}$$

$$\Rightarrow \frac{1}{\delta} = \frac{2-\alpha-\gamma}{2-\alpha+\gamma} \quad \text{(L116)}$$

$$\Rightarrow \delta = 1+\frac{\gamma}{\beta}, \quad \text{(L117)}$$

$$\langle \Delta N^2 \rangle = -\frac{k_B T N^2}{\mathcal{V}^2}\frac{\partial \mathcal{V}}{\partial P} = k_B T n^2 \mathcal{V} K_T \quad \text{(L118)}$$

$$= \left[\int d\vec{r}d\vec{r}'\, \langle n(\vec{r})n(\vec{r}')\rangle\right] - \langle N\rangle^2 \quad \text{(L119)}$$

$$= \mathcal{V}n\left\{1+n\int d\vec{r}(g(r)-1)\right\}. \quad \text{(L120)}$$

$$g(r) \sim \frac{e^{-r/\xi}}{r^{1+\eta}}, \quad \text{(L121)}$$

one has

$$K_T \sim \int d\vec{r}\, g(r). \tag{L122}$$

$$K_T \sim \xi^3 \xi^{-1-\eta} \int d\vec{s} \frac{e^{-s}}{s^{1+\eta}} \tag{L123}$$

$$\sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}. \tag{L124}$$

$$(2-\eta)\nu = \gamma, \tag{L125}$$

$$\frac{\mathcal{G}}{k_B T \mathcal{V}} \sim |t|^{2-\alpha} \sim \xi^{-3} \tag{L126}$$

$$\Rightarrow 2-\alpha = 3\nu, \tag{L127}$$

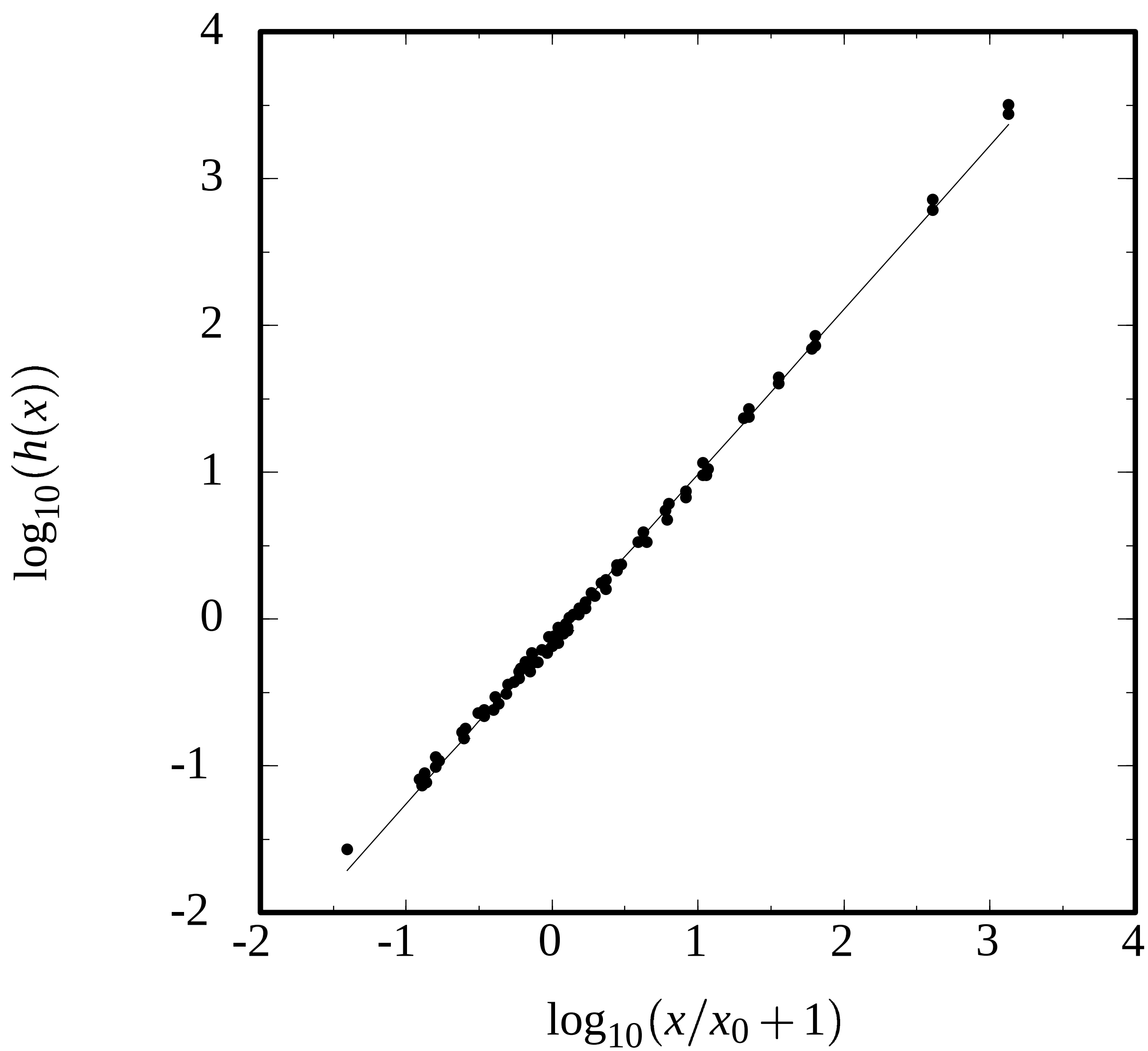


Figure 14: Scaling function $h = |H|/|M|^{\delta}$ versus $x = t/|M|^{1/\beta}$ [Source: Vicentini-Missoni (1972), p. 68.]