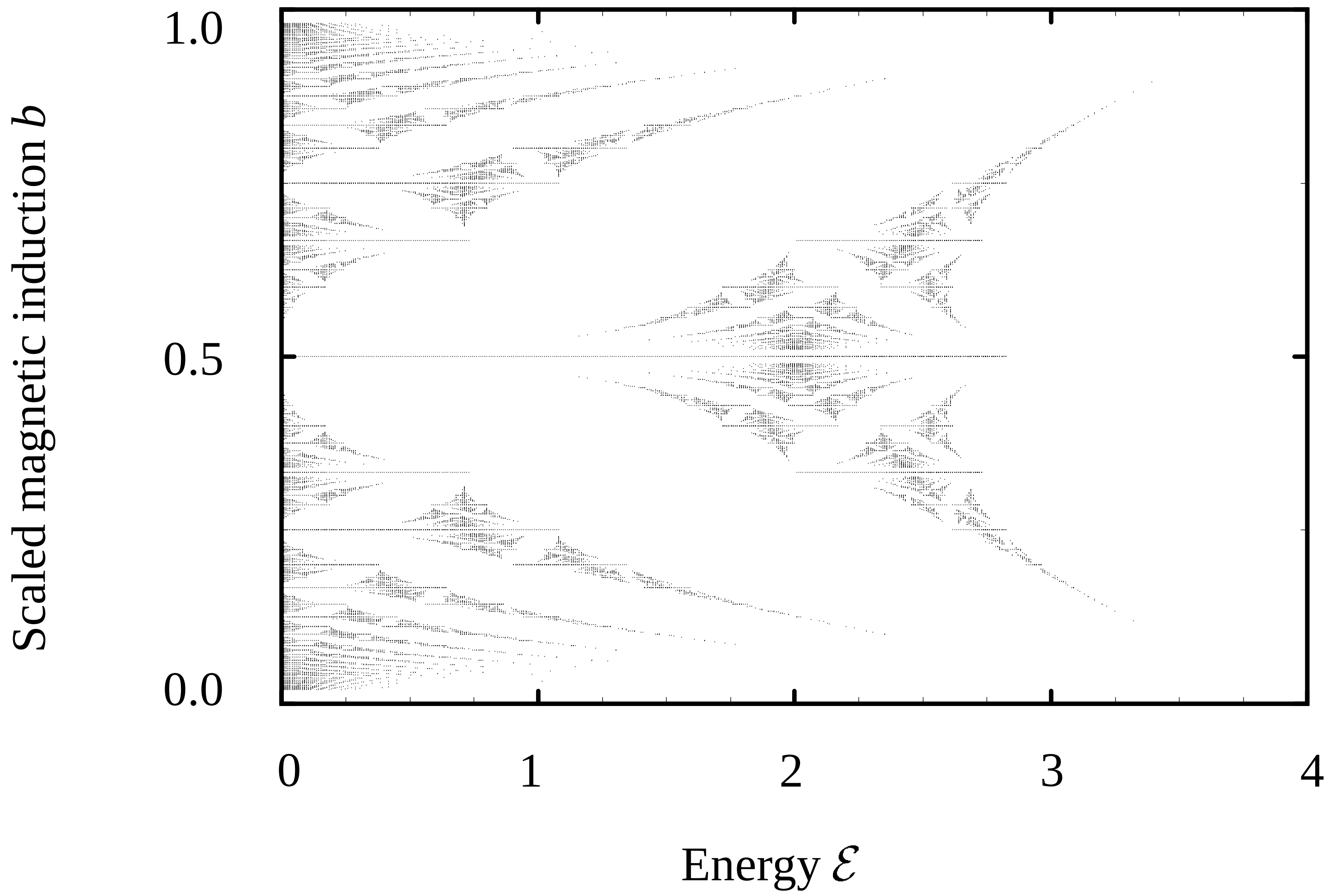
Magnetism of Ions and Electrons



Definitions

- Atomic Magnetism
- Hund's Rules
- Curie's Law
- Landau Diamagnetism
- Aharonov–Bohm Effect
- Hofstadter Butterfly
- Integer Quantum Hall Effect
- Fractional Quantum Hall Effect

$$\hat{\mathcal{H}} = \frac{1}{2m}\sum_l \left[\hat{P}_l + \frac{e}{c}\vec{A}(\hat{R}_l)\right]^2 + 2\mu_B B\hat{S}_l^z, \tag{L1}$$

$$\vec{A}(\hat{R}) = -\frac{1}{2}\hat{R}\times B\hat{z}, \tag{L2}$$

$$\hbar\hat{L} = \sum_j \hat{R}_j \times \hat{P}_j \tag{L3}$$

$$\Rightarrow \hat{P}_j\cdot\vec{A} = -\frac{1}{2}\hat{P}_j\cdot\hat{R}\times\vec{B} = \frac{1}{2}\vec{B}\cdot\hat{R}_j\times\hat{P}_j \tag{L4}$$

$$\Rightarrow \hat{\mathcal{H}} = \frac{1}{2m}\sum_l \hat{P}_l^2 + \mu_B\left(\hat{L}+2\hat{S}\right)\cdot\vec{B} + \frac{e^2}{8mc^2}B^2\sum_j\left(\hat{X}_j^2+\hat{Y}_j^2\right). \tag{L5}$$

$$\Delta\mathcal{E}_l = \mu_B\vec{B}\cdot\langle l|\hat{L}+2\hat{S}|l\rangle + \sum_{l'\neq l}\frac{|\langle l|\mu_B\vec{B}\cdot(\hat{L}+2\hat{S})|l'\rangle|^2}{\mathcal{E}_l-\mathcal{E}_{l'}} + \frac{e^2B^2}{8mc^2}\langle l|\sum_j\left(\hat{X}_j^2+\hat{Y}_j^2\right)|l\rangle. \tag{L6}$$

$$\mu_B B = 5.79\cdot 10^{-5}[B/\text{tesla}]\ \text{eV}. \tag{L7}$$

Hund's Rules

1. Maximize S
2. Maximize L, with electrons in different orbitals
3. Less than half full....

$$J = |L - S| \tag{L8a}$$

More than half full....

$$J = L + S \tag{L8b}$$

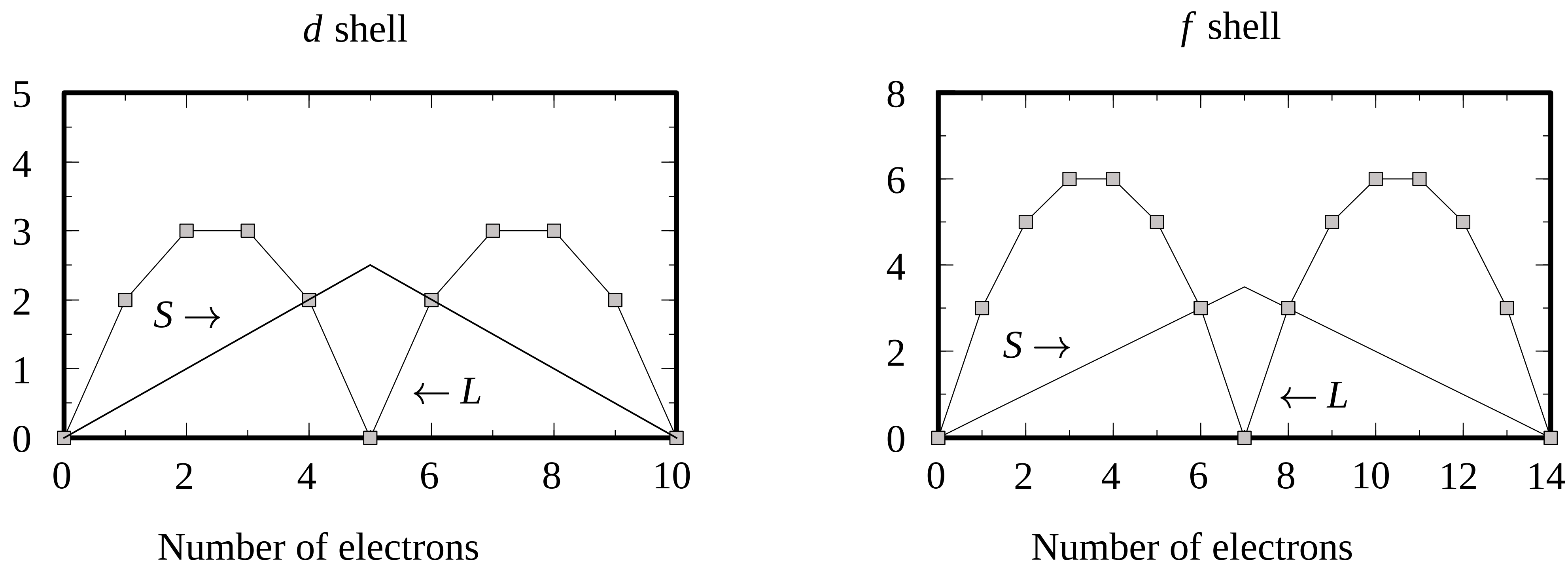


Figure 1: Hund's rules for *d* and *f* shells predict values for spin angular momentum *S* and orbital angular momentum *L* as indicated.

$$\langle l|\hat{L}_z + 2\hat{S}_z|l\rangle. \tag{L9}$$

$$\langle JLSJ_z|\hat{L}_z + 2\hat{S}_z|JLSJ_z'\rangle. \tag{L10}$$

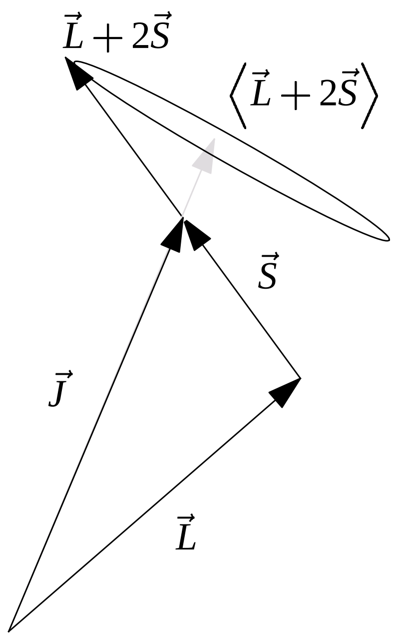


Figure 2: Expectation value of $\vec{L}+\vec{S}$ lies along $\vec{J}$.

$$\langle JLSJ_z|\hat{V}|JLSJ_z'\rangle = g(JLS)\langle JLSJ_z|\hat{J}|JLSJ_z'\rangle, \tag{L11}$$

Finding $g(JLS)$

$$\langle JLSJ_z|\hat{L}_z+2\hat{S}_z|JLSJ_z'\rangle = g(JLS)\langle JLSJ_z|\hat{J}_z|JLSJ_z'\rangle \quad \text{(L12)}$$

$$= g(JLS)J_z\delta_{J_zJ_z'}. \quad \text{(L13)}$$

$$\langle JLSJ_z|\hat{L}+2\hat{S}|JLSJ_z'\rangle = g(JLS)\langle JLSJ_z|\hat{J}|JLSJ_z'\rangle \quad \text{(L14)}$$

$$\Rightarrow \langle JLSJ_z|\hat{L}+2\hat{S}|J'L'S'J_z'\rangle = g(JLS)\langle JLSJ_z|\hat{J}|J'L'S'J_z'\rangle \quad \text{(L15)}$$

$$\Rightarrow \sum_{L'J'S'J_z'}\langle JLSJ_z|\hat{L}+2\hat{S}|J'L'S'J_z'\rangle \cdot \langle J'L'S'J_z'|\hat{J}|J''L''S''J_z''\rangle$$

$$= g(JLS)\sum_{L'J'S'J_z'}\langle JLSJ_z|\hat{J}|J'L'S'J_z'\rangle \cdot \langle J'L'S'J_z'|\hat{J}|J''L''S''J_z''\rangle. \quad \text{(L16)}$$

$$\langle JLSJ_z|(\hat{L}+2\hat{S})\cdot\hat{J}|JLSJ_z'\rangle = g(JLS)\langle JLSJ_z|\hat{J}^2|JLSJ_z'\rangle. \quad \text{(L17)}$$

$$\hat{S}^2 = (\hat{J}-\hat{L})^2 = \hat{J}^2+\hat{L}^2-2\hat{L}\cdot\hat{J} \quad \text{(L18)}$$

Finding $g(JLS)$

$$\hat{L}^2 = (\hat{J} - \hat{S})^2 = \hat{J}^2 + \hat{S}^2 - 2\hat{S}\cdot\hat{J}. \tag{L19}$$

$$g(JLS) = \frac{1}{2}\frac{[3J(J+1) - L(L+1) + S(S+1)]}{J(J+1)}. \tag{L20}$$

Energy level splittings in magnetic field are

$$\frac{\mu_B B}{2}\frac{[3J(J+1) - L(L+1) + S(S+1)]}{J(J+1)}. \tag{L21}$$

Curie's Law

$$Z_{\text{ion}} = \sum_{J_z=-J}^{J} e^{-\beta g \mu_B B J_z} \tag{L22}$$

$$= \frac{e^{\beta g \mu_B B (J+1/2)} - e^{-\beta g \mu_B B (J+1/2)}}{e^{\beta g \mu_B B/2} - e^{-\beta g \mu_B B/2}}. \tag{L23}$$

$$\mathcal{F} = -k_B T \ln Z_{\text{ion}} + \frac{1}{8\pi} \int d\vec{r}\, B^2. \tag{L24}$$

$$H = \frac{4\pi}{\mathcal{V}} \frac{\partial \mathcal{F}}{\partial B} \Rightarrow M = \frac{B}{4\pi} - \frac{1}{\mathcal{V}} \frac{\partial \mathcal{F}}{\partial B} \tag{L25}$$

$$\Rightarrow M = n k_B T \frac{\partial}{\partial B} \ln Z_{\text{ion}} \tag{L26}$$

$$= n \mu_B g J \mathcal{B}_J(\beta \mu_B g J B), \tag{L27}$$

where

$$\mathcal{B}_J(x) = ? \qquad\qquad ? \tag{L28}$$

$$\coth x \approx \frac{1}{x} + \frac{x}{3} + \ldots \quad \text{(L29)}$$

$$\Rightarrow \mathcal{B}_J = \frac{1}{3}\frac{J+1}{J}\beta\mu_B gJB, \quad \text{(L30)}$$

so that

$$M \approx ng^2\left(\mu_B\right)^2 \frac{B}{k_BT}\frac{J(J+1)}{3}. \quad \text{(L31)}$$

$$\chi = n\frac{1}{3k_BT}\mu_{\text{eff}}^2, \quad \text{(L32)}$$

$$\mu_{\text{eff}} = g(JLS)\sqrt{J(J+1)}.\mu_B \quad \text{(L33)}$$

$$\mu_{\text{exp}} = \sqrt{\frac{3k_BT\chi}{n}}. \quad \text{(L34)}$$

Element	Term	μ_{eff}, Eq. (L33) (μ_B)	μ_{exp}, Eq. (L34) (μ_B)
La^{3+}	$4f^0\ {}^1S$	0	Diamagnetic
Ce^{3+}	$4f^1\ {}^2F_{5/2}$	2.5	2.3
Pr^{3+}	$4f^2\ {}^3H_4$	3.6	3.4
Nd^{3+}	$4f3\ {}^4I_{9/2}$	3.6	3.5
Pm^{3+}	$4f^4\ {}^5I_4$	2.7	Radioactive
Sm^{3+}	$4f^5\ {}^6H_{5/2}$	0.9	1.6
Eu^{3+}	$4f^6\ {}^7F_0$	0	3.4
Gd^{3+}	$4f^7\ {}^8S_{7/2}$	7.9	7.9
Tb^{3+}	$4f^8\ {}^7F_6$	9.7	9.5
Dy^{3+}	$4f^9\ {}^6H_{15/2}$	10.6	10.4
Ho^{3+}	$4f^{10}\ {}^5I_8$	10.6	10.4
Er^{3+}	$4f^{11}\ {}^4I_{15/2}$	9.6	9.4
Tm^{3+}	$4f^{12}\ {}^3H_6$	7.6	7.1
Yb^{3+}	$4f^{13}\ {}^2F_{7/2}$	4.5	4.9
Lu^{3+}	$4f^{14}\ {}^1S$	0	0

Curie's Law

Element	Term	$\mu_{\rm eff}$, Eq. (L33) (μ_B)	$\mu_{\rm eff}$, $J = S$ (μ_B)	$\mu_{\rm exp}$, Eq. (L34) (μ_B)
Ti^{3+}	$3d^1\ {}^2D_{3/2}$	1.6	1.7	1.8
V^{3+}	$3d^2\ {}^3F_2$	1.6	2.8	2.7
Cr^{3+}	$3d^3\ {}^4F_{3/2}$	0.8	3.9	3.8
Mn^{3+}	$3d^4\ {}^5D_0$	0.0	4.9	4.9
Fe^{3+}	$3d^5\ {}^6S_{5/2}$	5.9	5.9	5.9
Fe^{2+}	$3d^6\ {}^5D_4$	6.7	4.9	5.3
Co^{2+}	$3d^7\ {}^4F_{9/2}$	6.5	3.9	4.0
Ni^{2+}	$3d^8\ {}^3F_4$	5.6	2.8	2.9–3.5
Cu^{2+}	$3d^9\ {}^2D_{5/2}$	3.6	1.7	1.7–1.9

$$\chi \approx -n\frac{e^2}{4mc^2}\,6\,\frac{2}{3}r^2. \tag{L35}$$

$$\Delta\mathcal{E} = \frac{e^2}{8mc^2}B^2\langle 0|\sum_j\left(\hat{X}_j^2+\hat{Y}_j^2\right)|0\rangle - \sum_{l'\neq 0}\frac{|\langle l'|\mu_B\vec{B}\cdot(\hat{L}+2\hat{S})|0\rangle|^2}{\mathcal{E}_{l'}-\mathcal{E}_0}. \tag{L36}$$

Element:	He	Ne	Ar	Kr	Xe
$-\chi$, experiment ($10^{-6}\,\text{cm}^3\,\text{mole}^{-1}$):	1.88	7.02	19.18	28.49	43.33
$-\chi$, Eq. (L35)$\times$0.35 ($10^{-6}\,\text{cm}^3\,\text{mole}^{-1}$):	0.99	14.82	20.54	23.74	27.95

Pauli Paramagnetism

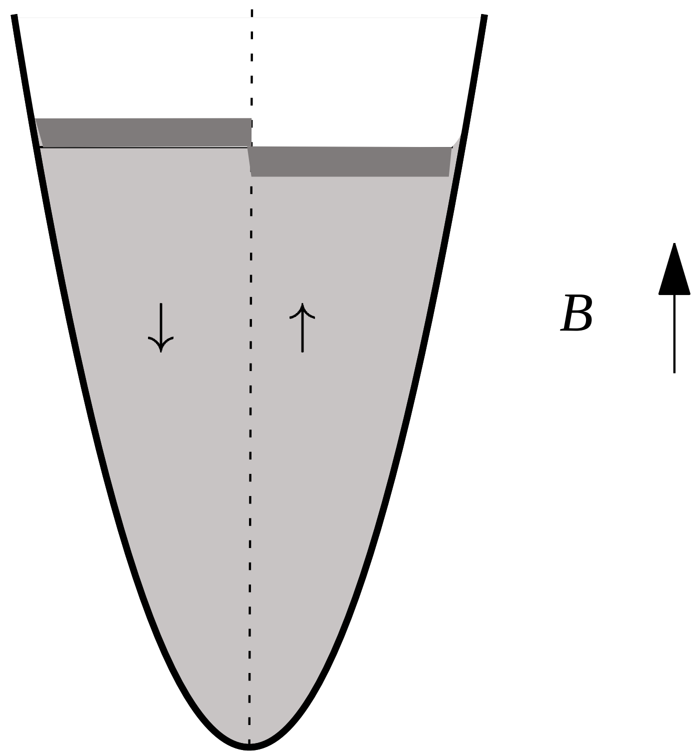


Figure 3: Pauli susceptibility

$$\mathcal{E}_{\vec{k}} = \mathcal{E}^0_{\vec{k}} + \mu_B B. \tag{L37}$$

$$N_{\text{up}} = \mathcal{V} \int d\mathcal{E}^0 \, \frac{D\left(\mathcal{E}^0\right)}{2} f(\mathcal{E}^0 + \mu_B B), \tag{L38}$$

Pauli Paramagnetism

$$N_{\text{down}} = \mathcal{V}\int d\mathcal{E}^0\,\frac{D\left(\mathcal{E}^0\right)}{2}\,f(\mathcal{E}^0-\mu_B B). \tag{L39}$$

$$N_{\text{up}} \approx \frac{N}{2}-\frac{\mu_B B}{2}\frac{\partial N}{\partial \mu}, \tag{L40}$$

$$N_{\text{down}} \approx \frac{N}{2}+\frac{\mu_B B}{2}\frac{\partial N}{\partial \mu}. \tag{L41}$$

$$M=\frac{\mu_B}{\mathcal{V}}\left(N_{\text{down}}-N_{\text{up}}\right)=? \qquad ? \tag{L42}$$

$$\chi=\frac{\partial M}{\partial H}\approx\frac{\partial M}{\partial B}=(\mu_B)^2\frac{1}{\mathcal{V}}\frac{\partial N}{\partial \mu}, \tag{L43}$$

$$\chi=\mu_B^2 D(\mathcal{E}_F). \tag{L44}$$

$$\chi=\frac{\mu_B^2 k_F m}{\pi^2\hbar^2}=4.757\cdot 10^{-7}\,(n/[10^{22}\cdot \text{cm}^{-3}])^{1/3}. \tag{L45}$$

Landau Diamagnetism

$$\omega_c = \frac{eB}{mc}, \tag{L46}$$

$$x_0 = \frac{-\hbar k_y}{m\omega_c}; \tag{L47}$$

$$\mathcal{E}_{\nu,k_z,k_y} = \frac{\hbar^2 k_z^2}{2m} + (\nu + \tfrac{1}{2})\hbar\omega_c. \tag{L48}$$

$$0 < x_0 < L \Rightarrow 0 < ? \qquad ? < L \tag{L49}$$

$$\Rightarrow 0 > l_2 > ? \qquad ?. \tag{L50}$$

$$\Rightarrow N = \frac{BA}{\Phi_0} = \frac{\Phi}{\Phi_0} \tag{L51}$$

$$\Phi_0 \equiv \frac{hc}{e} = 4.14 \cdot 10^{-7}\,\mathrm{G\,cm}^2; \tag{L52}$$

$$D(k_z,\nu) = 2\frac{m\omega_c}{2\pi\hbar}\frac{1}{2\pi}. \tag{L53}$$

$$D(\mathcal{E},\nu) = \frac{2}{(2\pi)^2}\frac{\hbar\omega_c}{2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[\mathcal{E}-\left(\nu+\tfrac{1}{2}\right)\hbar\omega_c\right]^{-1/2} \tag{L54}$$

$$\equiv \hbar\omega_c G\left[\mathcal{E}-\left(\nu+\tfrac{1}{2}\right)\hbar\omega_c\right], \tag{L55}$$

with

$$G(x) = \frac{1}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}x^{-1/2}. \tag{L56}$$

$$\Pi = -k_BT\mathcal{V}\int d\mathcal{E}\sum_\nu D(\mathcal{E},\nu)\ln[1+e^{\beta(\mu-\mathcal{E})}] \tag{L57}$$

$$= -k_BT\hbar\omega_c\mathcal{V}\int d\mathcal{E}\sum_{\nu=0}^{\infty}G(\mathcal{E})\ln[1+e^{\beta[\mu-(\mathcal{E}+(\nu+1/2)\hbar\omega_c)]}]. \tag{L58}$$

$$\sum_{\nu=0}^{\infty} F\left(\nu+\frac{1}{2}\right) \approx \int_0^{\infty} F(x)dx + \frac{1}{24}F'(0). \tag{L59}$$

$$\begin{aligned}\Pi = & -\mathcal{V}\int d\mathcal{E}\, k_B T\, \hbar\omega_c\, G(\mathcal{E}) \int d\nu \ln[1+e^{\beta\mu-\beta(\mathcal{E}+\nu\hbar\omega_c)}] \\ & + \frac{\mathcal{V}}{24}\int d\mathcal{E}\, (\hbar\omega_c)^2 G(\mathcal{E}) \frac{1}{e^{\beta\mathcal{E}-\beta\mu}+1} \end{aligned} \tag{L60}$$

$$= \Pi_0 + \frac{\mathcal{V}}{24}(\hbar\omega_c)^2 \int d\mathcal{E}\, G(\mathcal{E}) f(\mathcal{E}), \tag{L61}$$

with

$$\Pi_0 = -\mathcal{V}\int d\mathcal{E} \int_0^{\infty} dx\, k_B T G(\mathcal{E}) \ln\left[1+e^{\beta(\mu-\mathcal{E}-x)}\right]. \tag{L62}$$

$$\mathcal{V}\int d\mathcal{E}\, G(\mathcal{E}) f(\mathcal{E}) = -\frac{\partial^2 \Pi_0}{\partial \mu^2}. \tag{L63}$$

$$\Pi = \Pi_0 - \frac{1}{6}(B\mu_B)^2 \frac{\partial^2 \Pi_0}{\partial \mu^2} \tag{L64}$$

$$\Rightarrow M = -\frac{\partial \Pi}{\partial H}\Big|_\mu \approx -\frac{\partial \Pi}{\partial B}\Big|_\mu = -\frac{1}{3} B\mu_B^2 \frac{\partial N}{\partial \mu} \tag{L65}$$

$$\Rightarrow \chi = -\frac{1}{3}\mu_B^2 \frac{\partial N}{\partial \mu}. \tag{L66}$$

$$\chi = \frac{2}{3}\mu_B^2 \frac{\partial N}{\partial \mu} \tag{L67}$$

$$= \frac{2}{3} \frac{\mu_B^2 k_F m}{\pi^2 \hbar^2}. \tag{L68}$$

Landau Diamagnetism

Metal	Z	χ [Eq. (L68)] $(10^{-6}\,\mathrm{cm}^3\,\mathrm{mole}^{-1})$		χ (Experimental) $(10^{-6}\,\mathrm{cm}^3\,\mathrm{mole}^{-1})$
Li	1	6.90	p	25.00
Na	1	10.26	p	14.00
K	1	15.83	p	18.00
Au	1	5.84	d	−28.00
Be	2	4.50	d	−9.00
Mg	2	9.08	p	6.00
Ba	2	17.78	p	20.00
Zn	2	6.86	d	−9.15
Cd	2	8.66	d	−20.23
Hg	2	5.96	d	−17.10
Al	3	8.32	p	16.40
Ga	3	9.29	d	−21.68
Sn	4	12.65	d	−29.68
Bi	5	16.40	d	−271.67

$$\Phi = \int d^2 r\, B_z = \int d\vec{l} \cdot \vec{A}, \tag{L69}$$

$$A_\phi = \frac{\Phi}{2\pi r}. \tag{L70}$$

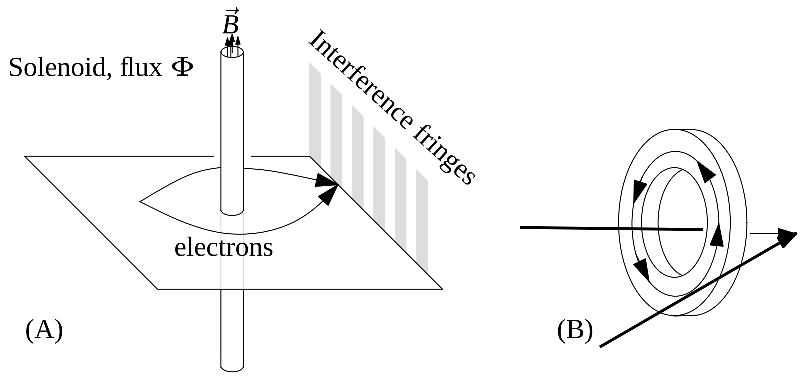
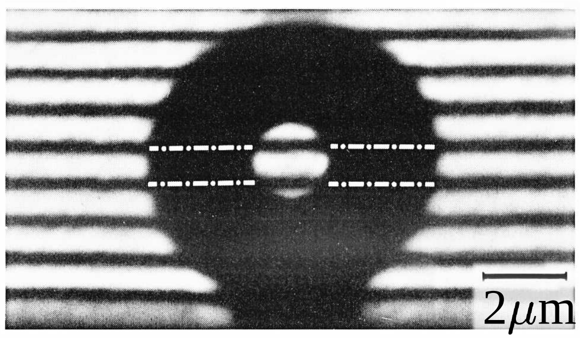


Figure 4: (A) Electrons traveling around a flux tube (B)Small toroidal magnet with no flux leakage

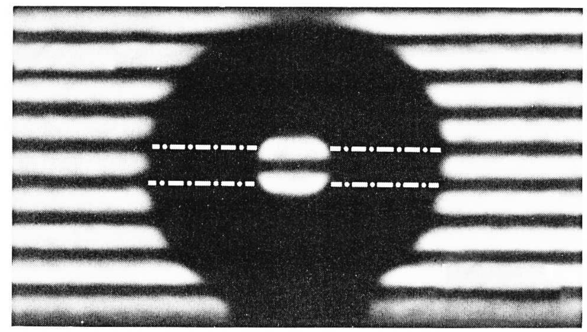
$$\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\vec{A}\right]^2 \psi = \mathcal{E}\psi \tag{L71}$$

Aharonov–Bohm Effect

$$\Rightarrow \psi \quad \propto \quad \exp\left[i\vec{k}\cdot\vec{r} - i\frac{e}{\hbar c}\int^{\vec{r}} d\vec{r}\,'\cdot\vec{A}(\vec{r}\,')\right]. \tag{L72}$$



(A)



(B)

Figure 5: Interference fringes of electrons passing through small toroidal magnet. In (A) the phase change is 0, while in (B) the phase change is π. [Source: Tonomura (1993), p. 67.]

$$\hat{P} - \frac{e}{c}\vec{A} = e^{ie\vec{A}\cdot\hat{R}/\hbar c}\hat{P}e^{-ie\vec{A}\cdot\hat{R}/\hbar c} \tag{L73}$$

$$\hat{\mathcal{H}} \to e^{ie\vec{A}\cdot\hat{R}/\hbar c}\hat{\mathcal{H}}e^{-ie\vec{A}\cdot\hat{R}/\hbar c}. \tag{L74}$$

$$\sum_{\vec{R}\vec{\delta}} e^{-ie\vec{A}\cdot\vec{\delta}/\hbar c}|\vec{R}\rangle \mathfrak{t}\langle\vec{R}+\vec{\delta}| + \sum_{\vec{R}} |\vec{R}\rangle U\langle\vec{R}|. \tag{L75}$$

$$2\psi_l \cos(2\pi lb - \kappa) + \psi_{l+1} + \psi_{l-1} = \mathcal{E}\psi_l, \tag{L76}$$

$$b = \frac{Ba^2}{\Phi_0} \tag{L77}$$

$$\kappa = ak_x. \tag{L78}$$

$$\psi_{l+q} = e^{ikq}\psi_l. \tag{L79}$$

$$\begin{pmatrix} \psi_{l+1} \\ \psi_l \end{pmatrix} = \begin{pmatrix} \mathcal{E} - 2\cos(2\pi lb - \kappa) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_l \\ \psi_{l-1} \end{pmatrix}. \tag{L80}$$

$$e^{iqk} \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} \psi_{q+1} \\ \psi_q \end{pmatrix} = \mathbf{Q}(\mathcal{E},\kappa) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} \tag{L81}$$

$$\mathbf{Q} = \prod_{l=1}^{q} \begin{pmatrix} \mathcal{E} - 2\cos(2\pi lb - \kappa) & -1 \\ 1 & 0 \end{pmatrix} \tag{L82}$$

$$\Rightarrow \mathrm{Det}\left| \mathbf{Q}(\mathcal{E},\kappa) - e^{iqk} \right| = 0. \tag{L83}$$

$$\mathrm{Det}\{\mathbf{Q}(\mathcal{E},\kappa)\} + e^{2iqk} - \mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa)\}\, e^{iqk} = 0. \tag{L84}$$

$$\mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa)\} = 2\cos qk, \tag{L85}$$

$$\mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa)\} = \mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa')\} \tag{L86}$$

$$\mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa)\} = \sum_{l=-\infty}^{\infty} F_l e^{iq\kappa l}. \tag{L87}$$

$$\mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa)\} = F_0(\mathcal{E}) + F_1(\mathcal{E})e^{iq\kappa} + F_1^*(\mathcal{E})e^{-iq\kappa}. \tag{L88}$$

$$\prod_{l=1}^{q}(-)\left[e^{i(2\pi lb-\kappa)} + e^{-i(2\pi lb-\kappa)}\right] \tag{L89}$$

$$\begin{aligned} F_1(\mathcal{E}) &= (-1)^q \prod_{l=1}^{q} e^{-2\pi ilb} && \text{(L90)} \\ &= (-1)^q e^{-2\pi biq(q+1)/2}. && \text{(L91)} \end{aligned}$$

$$F_0(\mathcal{E}) = \mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa_0)\}. \tag{L92}$$

$$(-1)^q 2\cos\left[2\pi b\left(q^2+q\right)/2 - q\kappa\right]. \tag{L93}$$

$$\pi p(q+1) - q\kappa, \tag{L94}$$

$$\kappa_0 = \frac{\pi}{2q}. \tag{L95}$$

$$\mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\kappa)\} = 2\cos qk = \mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\pi/2q)\} + 2\cos\left[\pi b(q^2+q) + \pi q - q\kappa\right]. \tag{L96}$$

$$\left|\mathrm{Tr}\{\mathbf{Q}(\mathcal{E},\pi/2q)\}\right| \le 4. \tag{L97}$$

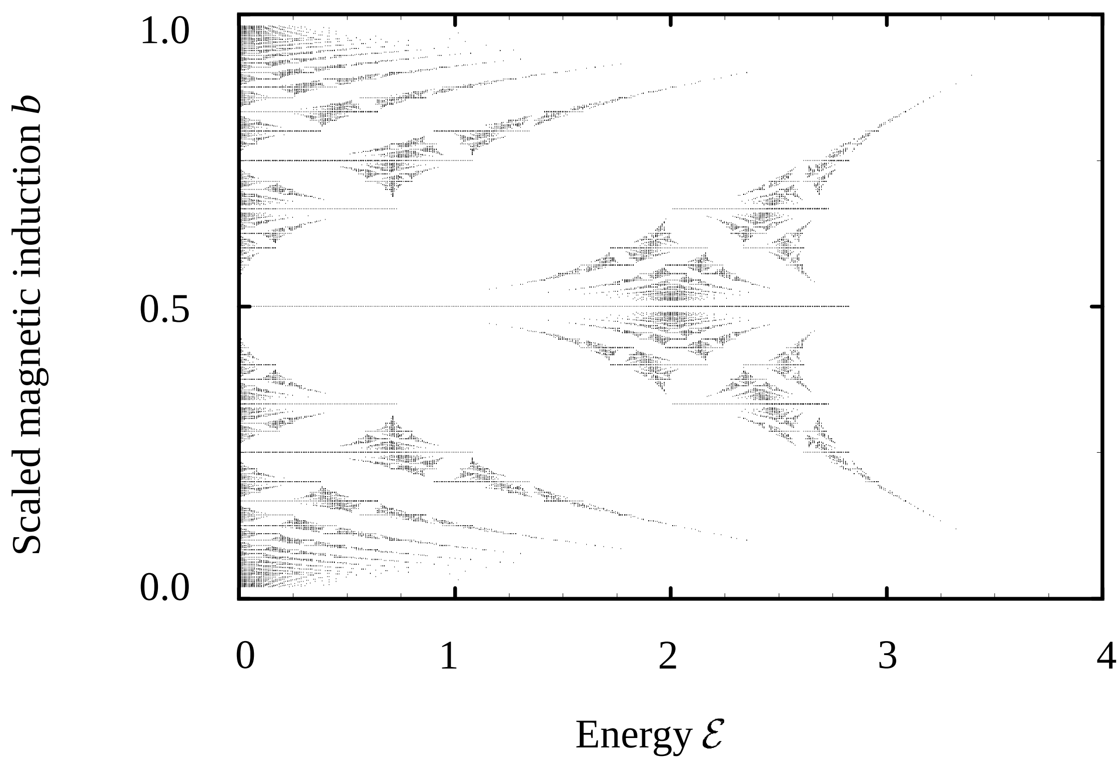


Figure 6: The Hofstadter Butterfly

$$\sigma_{xy} = \frac{\nu}{R_H}, \quad \text{(L98)}$$

$$R_H = \frac{h}{e^2} = 25\,813\,\Omega. \quad \text{(L99)}$$

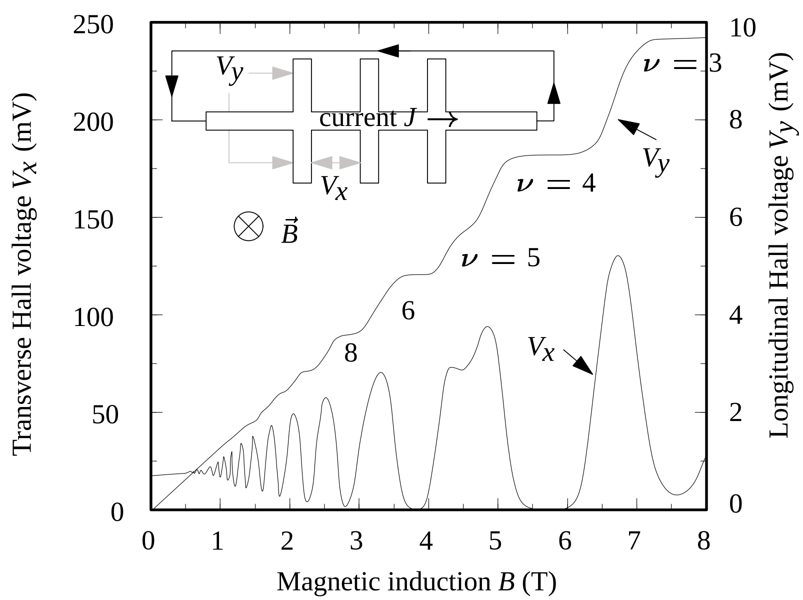


Figure 7: Integer quantum Hall effect. [Source: Cage (1987), p. 44.]

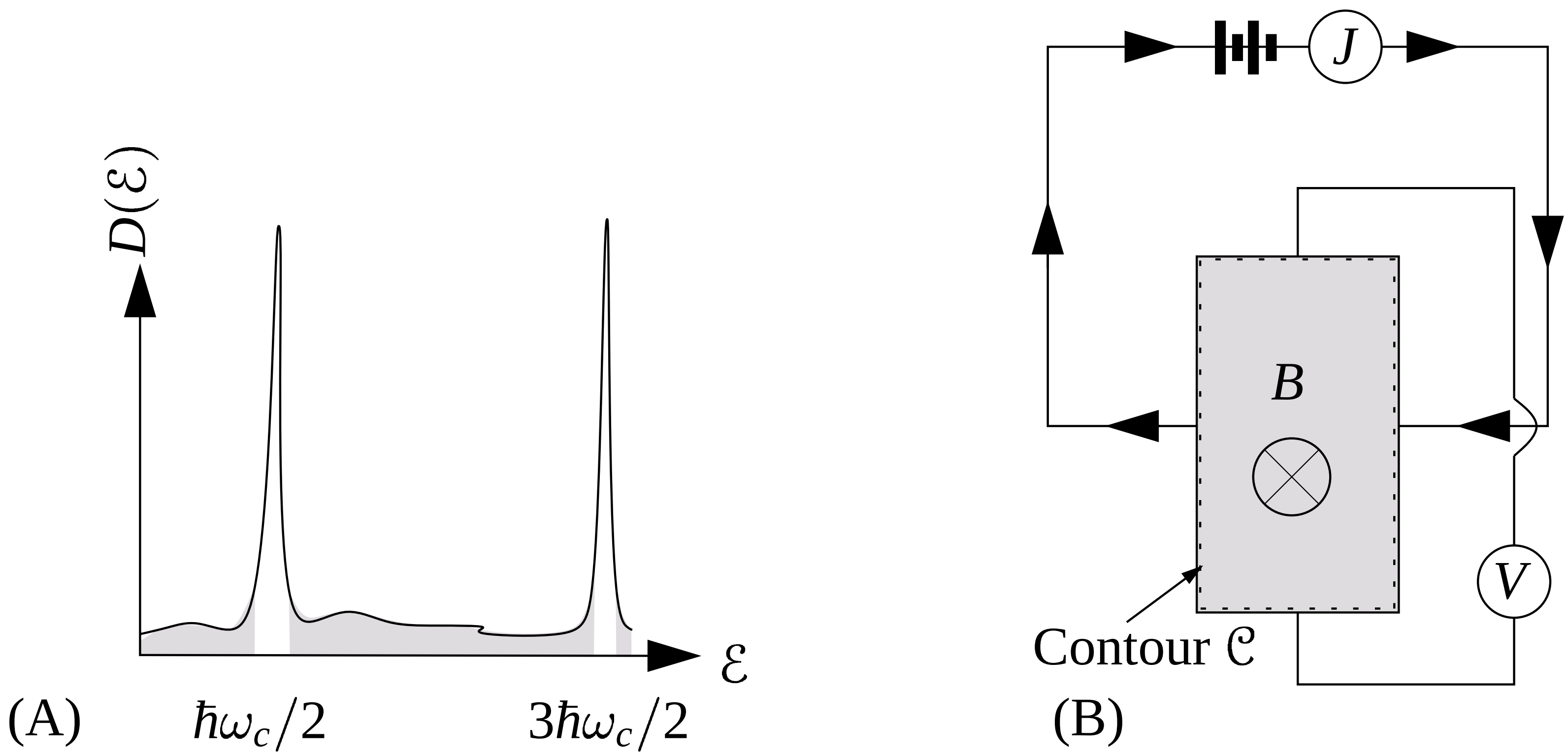


Figure 8: (A) States at energies $\hbar\omega_c(\nu+1/2)$,and localized states (shaded). (B) Schematic circuit for quantum Hall effect

Streda argument

$$\oint_{\mathcal{C}} d\vec{l} \cdot \vec{E} = \frac{-1}{c} \frac{\partial \Phi}{\partial t}. \tag{L100}$$

$$j_\perp = \sigma_{xy} E_{||}, \tag{L101}$$

$$\frac{1}{\sigma_{xy}} \oint_{\mathcal{C}} dl\, j_\perp = \frac{-1}{c} \frac{\partial \Phi}{\partial t} \tag{L102}$$

$$\Rightarrow \frac{\partial Q}{\partial t} = -\frac{\sigma_{xy}}{c} \frac{\partial \Phi}{\partial t} \tag{L103}$$

$$\Rightarrow \sigma_{xy} = -c \frac{\partial Q}{\partial \Phi}. \tag{L104}$$

$$Q = -e\nu \frac{\Phi}{\Phi_0} \tag{L105}$$

$$\Rightarrow \sigma_{xy} = \frac{ec\nu}{\Phi_0} = \frac{\nu e^2}{h} = \frac{\nu}{R_H}. \tag{L106}$$

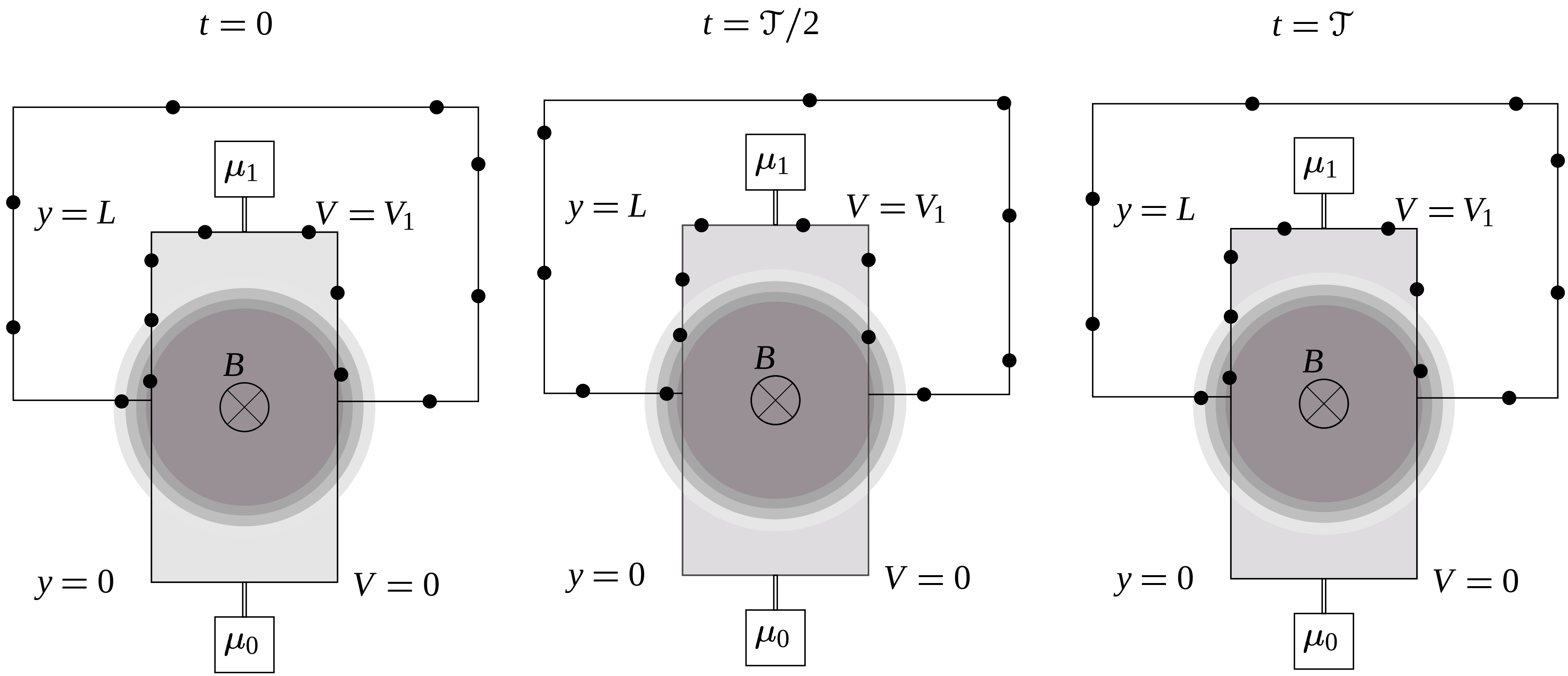


Figure 9: Gauge invariance for integer Hall effect

$$\vec{A} = \hat{y}xB, \tag{L107}$$

Laughlin's Argument

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial y}+\frac{exB}{c}\right)^2+U(\vec{r})-\mathcal{E}\right]\psi(\vec{r})=0. \tag{L108}$$

$$\mathcal{B}[\psi(x,L)]=0, \tag{L109}$$

$$\mathcal{B}[\psi^{\gamma}(x,L)e^{i\gamma}]=0. \tag{L110}$$

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial y}+\frac{exB}{c}+eE_y(\vec{r})t\right)^2+U(\vec{r})-\mathcal{E}\right]\psi(\vec{r})=0. \tag{L111}$$

$$\psi=e^{ieVt/\hbar}\tilde{\psi}, \tag{L112}$$

$$\vec{E}=-\vec{\nabla}V. \tag{L113}$$

$$\mathcal{B}[e^{-ietV_1/\hbar}\tilde{\psi}(x,L)] = 0. \tag{L114}$$

$$\gamma = -\frac{eV_1 t}{\hbar}. \tag{L115}$$

$$\frac{eV_1 t}{\hbar} = 2\pi. \tag{L116}$$

$$\mathcal{T} = \frac{h}{eV_1}. \tag{L117}$$

$$J_x = \frac{\nu e}{\mathcal{T}} = \frac{\nu e^2 V_1}{h}, \tag{L118}$$

$$\sigma_{xy} = \nu\frac{e^2}{h} = \frac{\nu}{R_H}. \tag{L119}$$

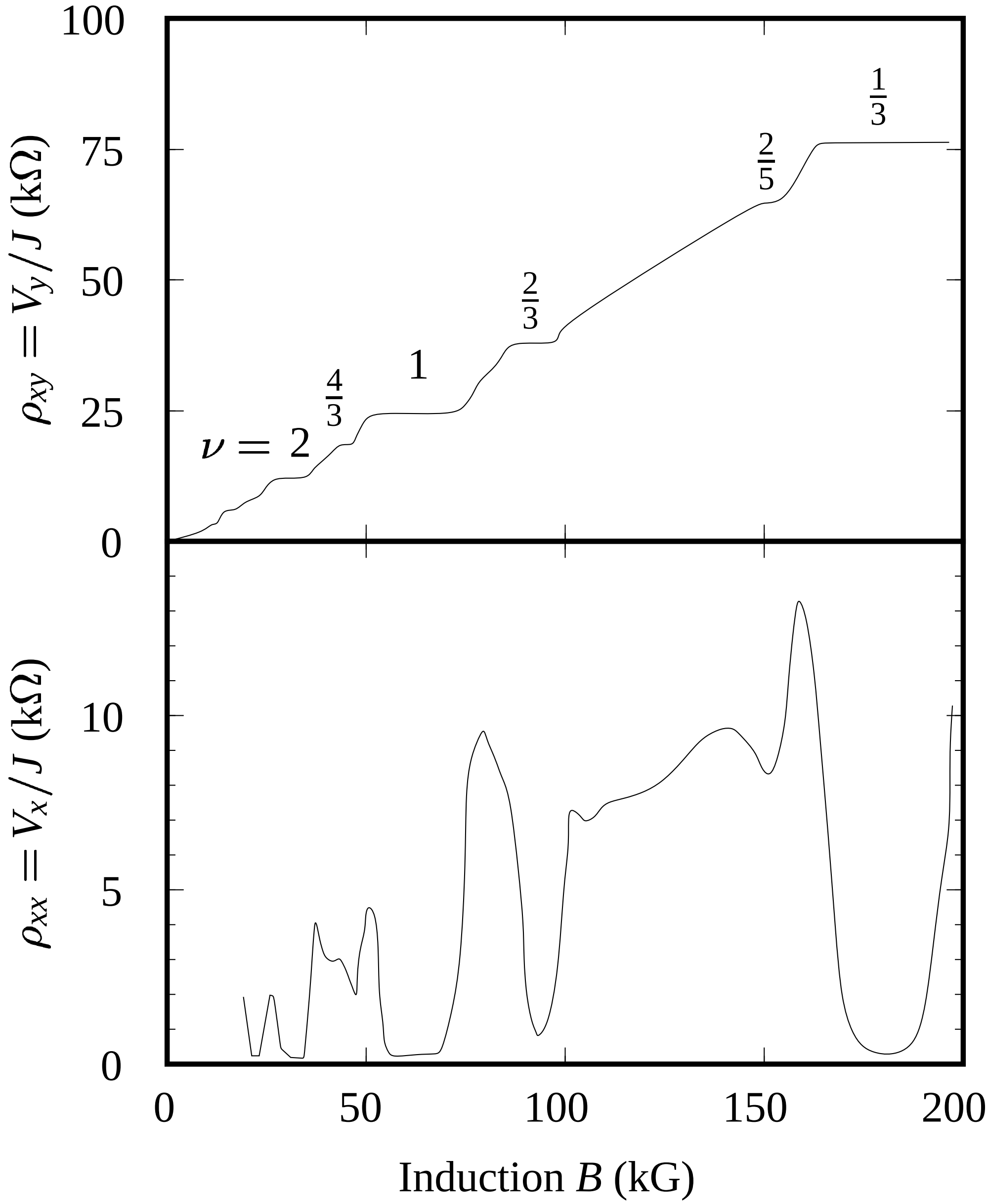


Figure 10: Fractional quantum Hall effect. Data of Boebinger, Chang, Störmer, and Tsui.

Laughlin's Wave Function

$$\frac{p}{q}\frac{e^2}{h}, \tag{L120}$$

$$\frac{e^2\sqrt{n}}{\epsilon^0\hbar\omega_c} = \frac{m^\star c e^2}{\epsilon^0\hbar\sqrt{eBhc}} = \frac{m^\star}{\epsilon^0 m}1.93\cdot 10^2/\sqrt{B/\mathrm{T}}. \tag{L121}$$

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - \frac{exB}{2c}\right)^2 + \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} + \frac{eyB}{2c}\right)^2 - \mathcal{E}\right]\psi(\vec{r}) = 0. \tag{L122}$$

$$l_B = \sqrt{\frac{2\hbar c}{eB}}, \text{ and define variables } \tilde{y} = \frac{y}{l_B} \text{ and } \tilde{x} = \frac{x}{l_B}. \tag{L123}$$

$$\frac{\hbar\omega_c}{4}\left[\left(\frac{1}{i}\frac{\partial}{\partial\tilde{y}} - \tilde{x}\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial\tilde{x}} + \tilde{y}\right)^2\right]\psi = \psi\mathcal{E}. \tag{L124}$$

$$z = \tilde{x} + i\tilde{y}, \text{ and define } \psi = e^{-|z|^2/2}\phi(z,\bar{z}). \tag{L125}$$

Laughlin's Wave Function

$$\hbar\omega_c \left\{ \frac{\partial \phi}{\partial \bar{z}} \bar{z} - \frac{\partial^2 \phi}{\partial z \partial \bar{z}} + \tfrac{1}{2}\phi \right\} = \mathcal{E}\phi. \quad \text{(L126)}$$

$$\phi(z,\bar{z}) = f(z) \Rightarrow \psi(z,\bar{z}) = f(z) e^{-|z|^2/2}, \quad \text{(L127)}$$

$$\Psi = f(z_0 \ldots z_{N-1}) e^{-\sum_{l=0}^{N-1} |z_l|^2/2}, \quad \text{(L128)}$$

$$\Psi = \prod_{l<l'} f_2(z_l - z_{l'}) e^{-\sum_{l=0}^{N-1} |z_l|^2/2}. \quad \text{(L129)}$$

$$\hat{L}_l = -i \frac{\partial}{\partial \theta_l} = \left[z_l \frac{\partial}{\partial z_l} - \bar{z}_l \frac{\partial}{\partial \bar{z}_l} \right], \quad \text{(L130)}$$

$$z \frac{\partial f_2(z)}{\partial z} = q f_2(z) \Rightarrow f_2(z) = z^q. \quad \text{(L131)}$$

$$\Psi = \prod_{l<l'} (z_l - z_{l'})^q e^{-\sum_{l=0}^{N-1} |z_l|^2/2}. \quad \text{(L132)}$$

Laughlin's Wave Function

$$\Psi = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ z_0 & z_1 & \ldots & z_{N-1} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ z_0^{N-1} & z_1^{N-1} & \ldots & z_{N-1}^{N-1} \end{vmatrix} e^{-\sum_{l=0}^{N-1} |z_l|^2/2}. \qquad \text{(L133)}$$

$$z_2^m - z_1^m = (z_2 - z_1) \sum_{l=0}^{m-1} z_2^l z_1^{m-l-1}. \qquad \text{(L134)}$$

$$e^{-|z|^2/2}, ze^{-|z|^2/2}, z^2 e^{-|z|^2/2} \ldots z^{N-1} e^{-|z|^2/2} \qquad \text{(L135)}$$

$$A = \pi N l_B^2 = \frac{2\pi N \hbar c}{eB} \Rightarrow N = \frac{BA}{\Phi_0}. \qquad \text{(L136)}$$

$$|z_0|^2 = q(N-1) \Rightarrow N = \frac{BA}{q\Phi_0}. \qquad \text{(L137)}$$

Fractional Charge

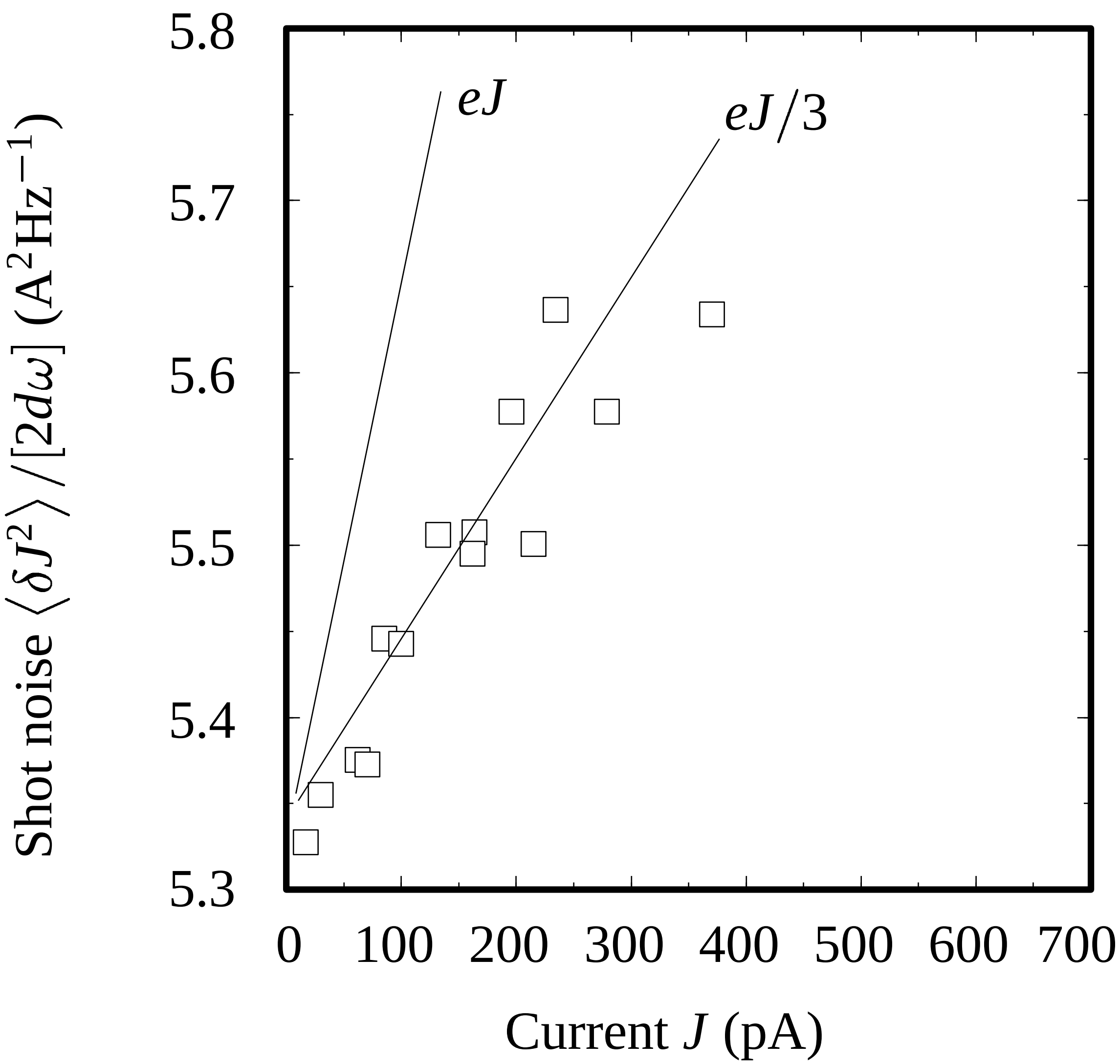


Figure 11: Shot noise for fractional quantum Hall effect [Source: Saminadayar et al. (1997), p. 2528.]