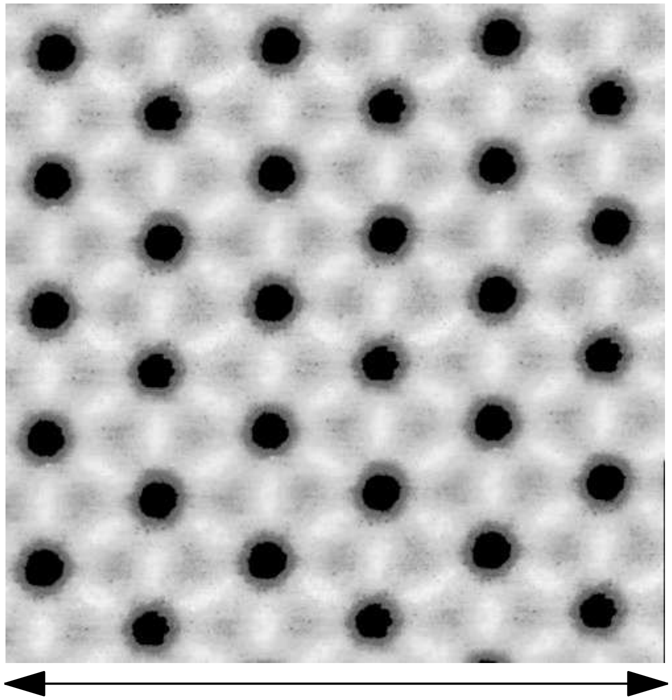


(B)

Definitions

- ☞ Perfect Diamagnetism
- ☞ Landau–Ginzburg Equations
- ☞ Type I and Type II Superconductors
- ☞ Flux Quantization
- ☞ Josephson Effect
- ☞ Superconducting Quantum Interference Devices (SQUIDS)
- ☞ Isotope Effect and Fröhlich Hamiltonian
- ☞ Cooper Problem
- ☞ Bardeen Cooper Schrieffer (BCS) Theory
- ☞ Bogoliubov Theory
- ☞ High-Temperature Superconductors

Expulsion of magnetic fields, not infinite conductivity, is the key.

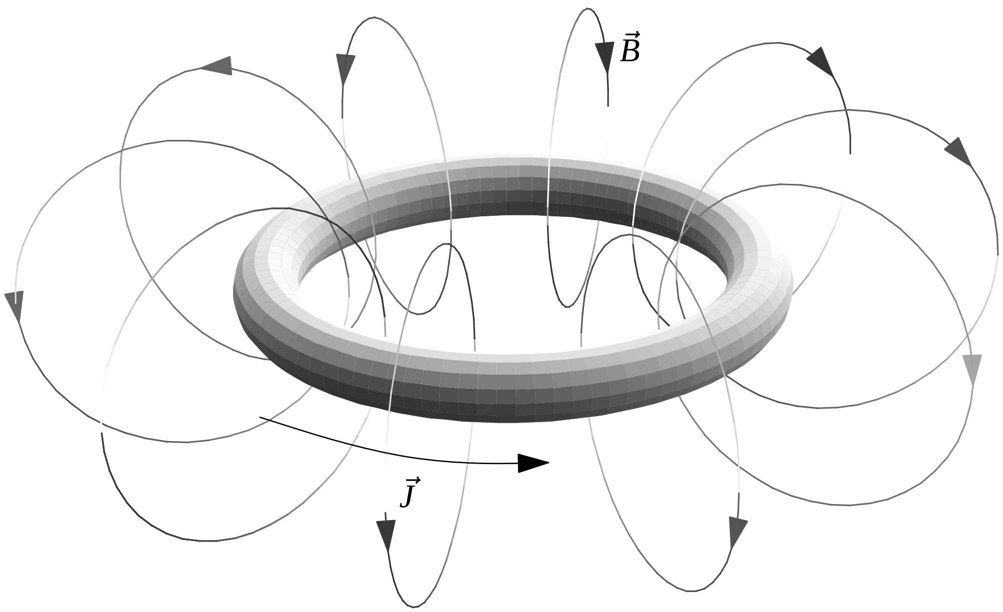


Figure 1: Flux threading a current loop

Wave function is rigid

$$
\begin{aligned}
m\dot{\vec{v}} &= -e\vec{E} && \text{(L1)}\\
\Rightarrow \frac{\partial \vec{j}}{\partial t} &= \frac{ne^2}{m}\vec{E} && \text{(L2)}\\
\Rightarrow \frac{\partial}{\partial t}\vec{\nabla}\times\vec{\nabla}\times\frac{\vec{B}}{\mu} &= \frac{4\pi ne^2}{mc}\vec{\nabla}\times\vec{E} && \text{(L3)}\\
\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\frac{\vec{B}}{\mu} &= -\frac{4\pi ne^2}{mc^2}(\vec{B}-\vec{B}_0). && \text{(L4)}
\end{aligned}
$$

London Penetration

$$\vec{B} + \lambda_L^2 \vec{\nabla} \times \vec{\nabla} \times \vec{B} = 0 \tag{L5}$$

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi \mu n e^2}}. \tag{L6}$$

$$\vec{B} + \lambda_L^2 \left(\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} \right) = 0. \tag{L7}$$

$$B_z = 0. \tag{L8}$$

$$B_x = \lambda_L^2 \frac{\partial^2 B_x}{\partial z^2} \;\Rightarrow\; B_x \propto e^{-z/\lambda_L}. \tag{L9}$$

$$\mathfrak{F} = \int d\vec{r}d\vec{r}' \sum_{\alpha\beta} \tfrac{1}{2} A_\alpha(\vec{r}) G_{\alpha\beta}(\vec{r}-\vec{r}') A_\beta(\vec{r}') + \delta(\vec{r}-\vec{r}') \frac{1}{8\pi} \vec{B}(\vec{r}) \cdot \vec{B}(\vec{r}'). \qquad \text{(L10)}$$

$$[\vec{\nabla} \times \vec{\nabla} \times \frac{\vec{A}(\vec{r})}{4\pi}]_\alpha = -\int d\vec{r}' \sum_\beta G_{\alpha\beta}(\vec{r}-\vec{r}') A_\beta(\vec{r}') \qquad \text{(L11)}$$

$$\Rightarrow j_\alpha(\vec{r}) = \frac{c}{4\pi}[\vec{\nabla} \times \vec{B}]_\alpha = -c\int d\vec{r}' \sum_\beta G_{\alpha\beta}(\vec{r}-\vec{r}') A_\beta(\vec{r}'). \qquad \text{(L12)}$$

$$\sum_\beta \left\{ G_{\alpha\beta}(\vec{k}) + \frac{1}{4\pi}(k^2\delta_{\alpha\beta} - k_\alpha k_\beta) \right\} A_\beta = 0. \qquad \text{(L13)}$$

$$G_{\alpha\beta} \to \left(\frac{1}{\mu} - 1\right) \frac{1}{4\pi} \left[k^2\delta_{\alpha\beta} - k_\alpha k_\beta\right]. \qquad \text{(L14)}$$

Phenomenological Free Energy

$$\frac{1/\mu - 1}{8\pi}\int d\vec{r}d\vec{r}'\,\delta(\vec{r}-\vec{r}')\vec{A}\cdot\vec{\nabla}\times\vec{\nabla}\times\vec{A} \tag{L15}$$

$$= \frac{1/\mu - 1}{8\pi}\int d\vec{r}\,B(\vec{r})^2. \tag{L16}$$

$$\mathcal{F} = \frac{1}{8\pi\mu}\int d\vec{r}\,B^2(\vec{r}), \tag{L17}$$

$$\lim_{k\to 0} G_{\alpha\beta} = \frac{1}{4\pi\lambda_L^2}\delta_{\alpha\beta}. \tag{L18}$$

$$\frac{1}{8\pi}\int d\vec{r}\,\frac{1}{\lambda_L^2}A^2(\vec{r}) + |\vec{\nabla}\times\vec{A}|^2, \tag{L19}$$

$$\frac{1}{\lambda_L^2}\vec{A} + \vec{\nabla}\times\vec{\nabla}\times\vec{A} = 0. \tag{L20}$$

$$\vec{B} + \lambda_L^2\,\vec{\nabla}\times\vec{\nabla}\times\vec{B} = 0, \tag{L21}$$

$$\mathfrak{F} = \mathfrak{F}_{\text{normal}} + \frac{1}{8\pi\mu} B_c^2. \tag{L22}$$

$$\tilde{\mathfrak{G}} \quad = \quad \mathfrak{F} - \vec{B} \cdot \frac{\delta \mathfrak{F}}{\delta \vec{B}} = \mathfrak{F}_{\text{normal}} - \frac{1}{8\pi\mu} B_c^2. \tag{L23}$$

$$\Delta \mathfrak{F} \equiv \mathfrak{F}_{\text{normal}} - \mathfrak{F}_{\text{superconducting}} = \frac{B_c^2}{8\pi\mu}. \tag{L24}$$

$$\Delta \mathfrak{F} = \frac{H_c^2}{8\pi}. \tag{L25}$$

$$\Delta S = \frac{\partial}{\partial T} \Delta \mathfrak{F} = \frac{H_c}{4\pi} \frac{\partial H_c}{\partial T}. \tag{L26}$$

$$\mathcal{F} = \int \frac{d\vec{r}}{\mathcal{V}} \alpha |\Psi|^2 + \frac{\beta}{2}|\Psi^4| + \frac{1}{8\pi}B^2 + \frac{1}{2m^\star}\left|\left[\frac{\hbar}{i}\vec{\nabla} + \frac{2e}{c}\vec{A}(\vec{r})\right]\Psi(\vec{r})\right|^2. \qquad \text{(L27)}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}, \qquad \text{(L28)}$$

$$\vec{j}(\vec{r}) \quad = \quad -\frac{2e\hbar}{2im^\star}[\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*] - \frac{4e^2}{m^\star c}\vec{A}\Psi^*\Psi. \qquad \text{(L29a)}$$

Minimizing with respect to Ψ^* leads to

$$0 \quad = \quad \left[\alpha + \beta|\Psi|^2 + \frac{1}{2m^\star}\left(\frac{\hbar}{i}\vec{\nabla} + \frac{2e}{c}\vec{A}\right)^2\right]\Psi. \qquad \text{(L29b)}$$

$$\hat{n} \cdot \left(\frac{\hbar}{i}\vec{\nabla} + \frac{2e}{c}\vec{A}\right)\Psi = 0. \qquad \text{(L30)}$$

Compare the following lengths:

$$\xi^2 = \frac{\hbar^2}{2m^\star|\alpha|}. \tag{L31}$$

$$\lambda_L^2 = \frac{m^\star c^2 \beta}{4\pi|\alpha|(2e)^2}. \tag{L32}$$

Type I and Type II Superconductors

Compound	T_c (K)	H_c (G)	ξ (Å)	λ_L (Å)
Al	1.18	105	13 000–16 000	160–500
Ba (P =20 GPa)	5.3			
Bi (P =8 GPa)	8.55			
Ce (P =5 GPa)	1.7			
Ga	1.09	58.9		
Hg	3.95	340		380–450
Ir	0.10	20.1		
Lu	0.1			
Mo	0.92	98		
P (P =17 GPa)	5.8			
Pb	7.20	803	510–960	390–630
Si (P =12 GPa)	7.1			
Sn	3.7	308	1 000–3 000	340–750
Te (P = 8 GPa)	4.3			
Th	1.37	162		
Ti	0.42	56		
Tl	2.4	180	4200	
U	1.8			
W	0.02	1.07		
Zn	0.85	52		
Zr	0.53	47		
Nb_3Sn	18.5	28	34	1 600
$YBa_2Cu_3O_{7-x}$	92	500	4–8	900–8 000
$HgBa_2Ca_2Cu_3O_y$	135			

$$|\Psi|^2 = \begin{cases} \Psi_0^2 \equiv -\dfrac{\alpha}{\beta} & \text{or} \\ 0. & \end{cases} \tag{L33}$$

$$\frac{\mathcal{F}}{\mathcal{V}} = -\frac{\alpha^2}{2\beta} \tag{L34}$$

$$H_c^2 = \frac{4\pi\alpha^2}{\beta}. \tag{L35}$$

$$\psi = \frac{\Psi}{\Psi_0}, \tag{L36}$$

$$-\xi^2\nabla^2\psi - \psi + \psi|\psi|^2 = 0, \tag{L37}$$

$$-\xi^2\psi'' - \psi + \psi^3 = 0. \tag{L38}$$

$$-\xi^2(\psi')^2 - \psi^2 + \frac{1}{2}\psi^4 = \text{ Const.} \tag{L39}$$

$$\psi' = \frac{1}{\sqrt{2}\xi}(1-\psi^2) \tag{L40}$$

$$\psi = \tanh\frac{x}{\sqrt{2}\xi}. \tag{L41}$$

$$\vec{j} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = -\frac{4e^2}{m^\star c}\Psi_0^2\vec{A}. \tag{L42}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = -\frac{4\pi}{c}\frac{4e^2}{m^\star c}\Psi_0^2\vec{B} = -\lambda_L^{-2}\vec{B}. \tag{L43}$$

$$\kappa = \lambda_L/\xi = \frac{m^\star c}{e\hbar}\sqrt{\frac{\beta}{8\pi}} \tag{L44}$$

$$\vec{a} = \frac{4e\vec{A}}{c\sqrt{2m^\star|\alpha|}} \tag{L45}$$

$$\psi - \psi|\psi|^2 - (-i\vec{\nabla} + \vec{a}/2)^2\psi = 0 \tag{L46}$$

$$\frac{\lambda_L^2}{\xi^2}\vec{\nabla}\times\vec{\nabla}\times\vec{a} = -\frac{1}{i}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) - |\psi|^2\vec{a}. \tag{L47}$$

$$\frac{1}{2m^\star}(-i\hbar\vec{\nabla} + \frac{2e\vec{A}}{c})^2\Psi = -\alpha\Psi. \tag{L48}$$

$$\omega_c = \frac{2eH_{c_2}}{m^\star c}, \tag{L49}$$

$$-\alpha = |\alpha| = \frac{e\hbar H_{c_2}}{m^\star c}. \tag{L50}$$

$$\frac{H_{c_2}}{H_c} = \sqrt{2}\kappa. \tag{L51}$$

$$\frac{\tilde{\mathcal{G}}}{A} = \frac{H_c^2}{4\pi}\sqrt{2}\xi\frac{2}{3}, \tag{L52}$$

$$\frac{\tilde{\mathcal{G}}}{A} = -\frac{H_c^2}{8\pi}\lambda_L; \tag{L53}$$

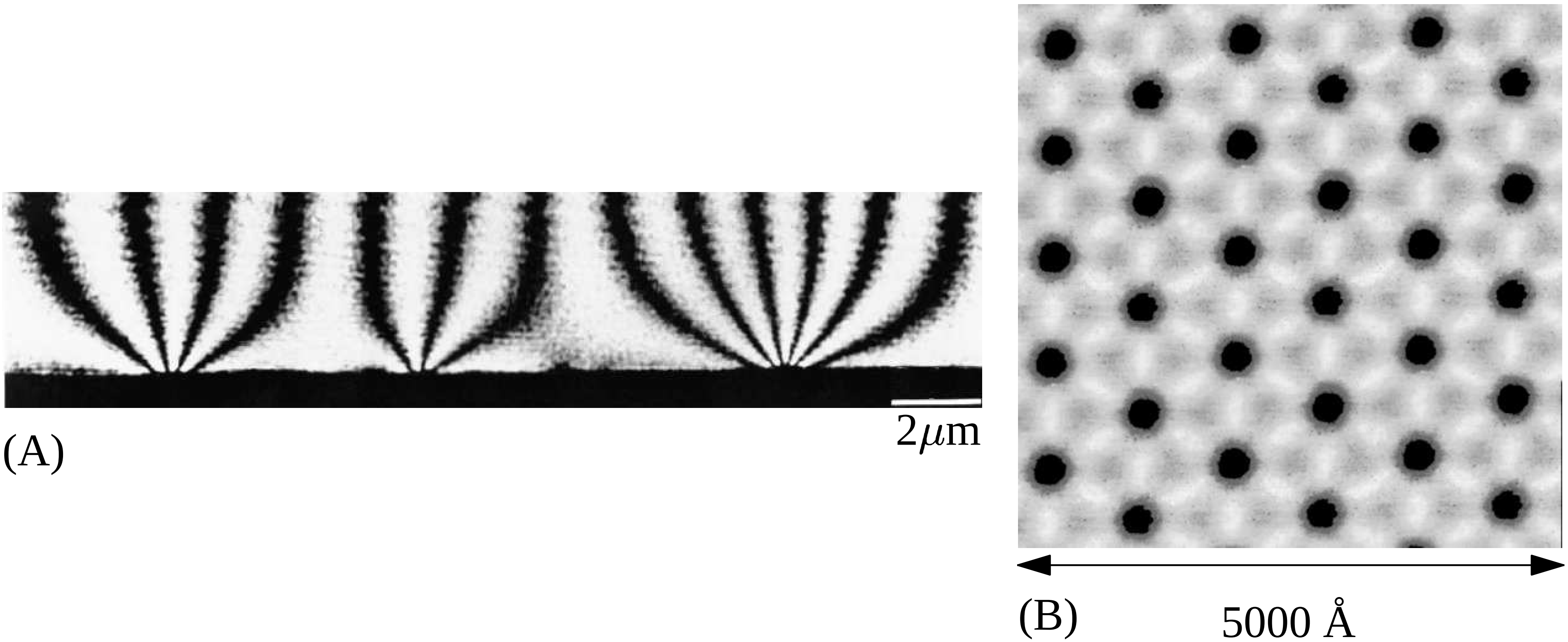


Figure 2: A Type II superconductor is unstable to the formation of flux tubes (A) Magnetic flux entering a lead film [Tonomura et al. (1986)] (B) Top view of an Abrikosov lattice of flux tubes in $NbSe_2$ [S. Pan and A. de Lozanne]

Flux Quantization

$$\vec{j} = -\frac{e^\star \hbar}{2im^\star}[\Psi^* \vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*] - \frac{e^{\star 2}}{m^\star c}\vec{A}\Psi^*\Psi. \quad \text{(L54)}$$

$$\Psi(\vec{r}) = \Psi_0 e^{i\phi(\vec{r})} \quad \text{(L55)}$$

$$\vec{j} = -\frac{\Psi_0^2}{m^\star}\left(\frac{e^{\star 2}}{c}\vec{A} + e^\star\hbar\vec{\nabla}\phi\right) \quad \text{(L56)}$$

$$\Rightarrow -\vec{\nabla}\phi = \frac{1}{\hbar}\left(\frac{m^\star}{e^\star\Psi_0^2}\vec{j} + \frac{e^\star}{c}\vec{A}\right). \quad \text{(L57)}$$

$$-\int d\vec{s}\cdot\vec{\nabla}\phi = 2\pi l. \quad \text{(L58)}$$

$$\int d\vec{s}\cdot\frac{1}{\hbar}\left[\frac{m^\star}{e^\star\Psi_0^2}\vec{j} + \frac{e^\star}{c}\vec{A}\right] = 2\pi l. \quad \text{(L59)}$$

$$\frac{e^\star}{c\hbar}\int d\vec{s}\cdot\vec{A} \quad = \quad 2\pi l \tag{L60}$$

$$\Rightarrow \int d^2r\,B_z = \Phi = \frac{2\pi l\hbar c}{e^\star} \quad = \quad l\frac{e}{e^\star}\Phi_0. \tag{L61}$$

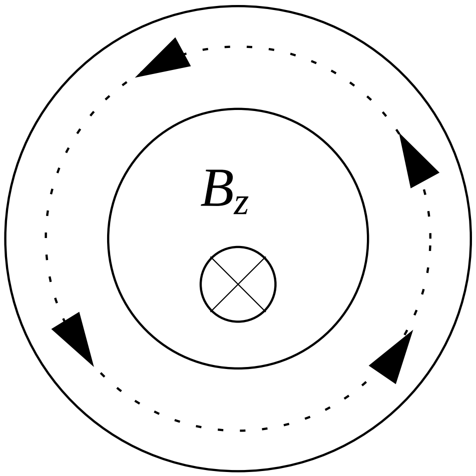


Figure 3: Magnetic flux that pierces a superconducting ring is quantized in units of $\Phi_0/2$.

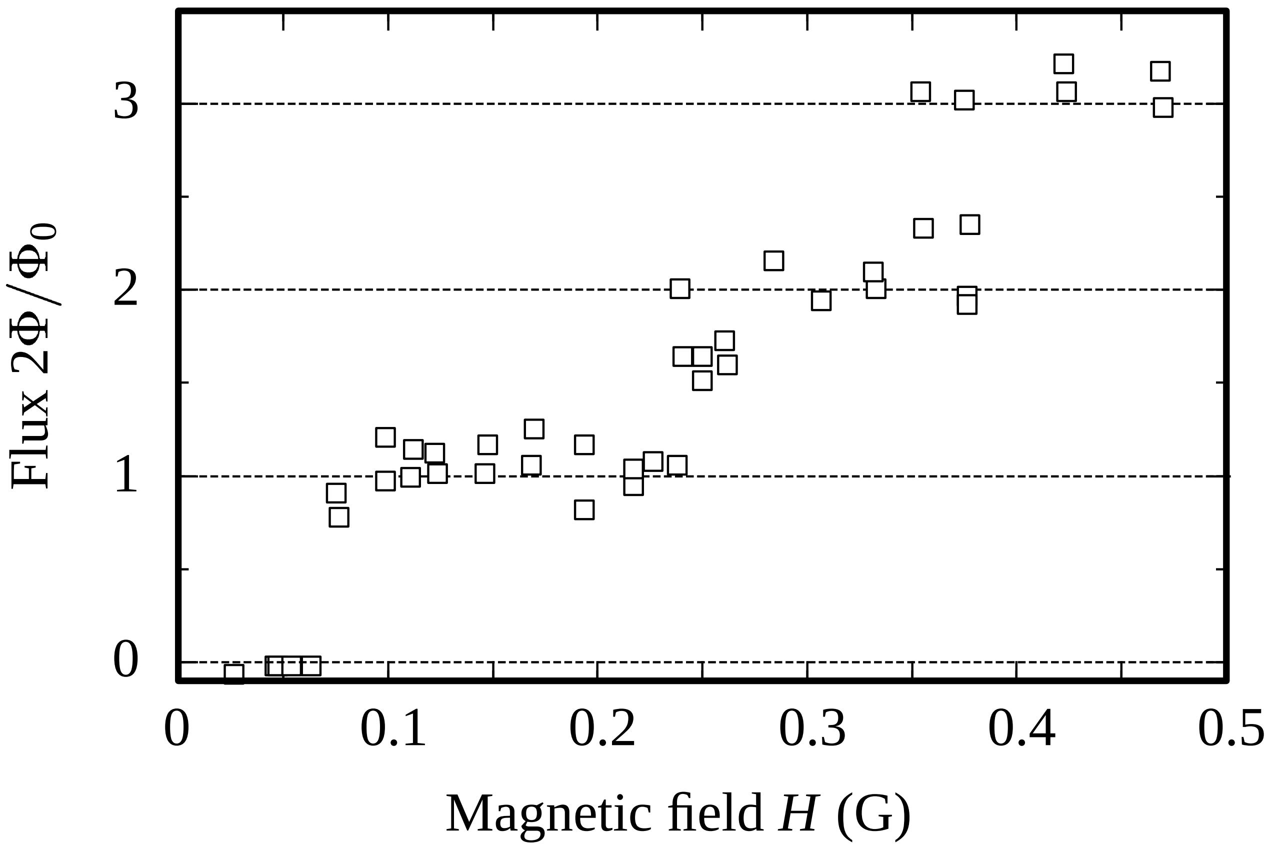


Figure 4: Trapped magnetic flux in a superconducting cylinder as a function of applied field. [Deaver and Fairbank (1961)]

$$\int d\vec{r}\, U(\vec{r})\left(\Psi_1^*(\vec{r})\Psi_2(\vec{r}) + \Psi_1(\vec{r})\Psi_2^*(\vec{r})\right) \tag{L62}$$

$$= \quad \epsilon\left(\Psi_1^*\Psi_2 + \Psi_1\Psi_2^*\right), \tag{L63}$$

$$\frac{\partial \Psi_1}{\partial t} = \frac{-i}{\hbar}\left[\mathcal{E}_1\Psi_1 + \epsilon\Psi_2\right] \tag{L64a}$$

$$\frac{\partial \Psi_2}{\partial t} = \frac{-i}{\hbar}\left[\mathcal{E}_2\Psi_2 + \epsilon\Psi_1\right]. \tag{L64b}$$

$$\Psi_l = \sqrt{n_l}\, e^{i\phi_l} \tag{L65}$$

$$\left(\frac{1}{2}\frac{\dot{n}_1}{\sqrt{n_1}} + i\sqrt{n_1}\dot{\phi}_1\right)e^{i\phi_1} = \frac{-i}{\hbar}\left[\mathcal{E}_1\sqrt{n_1}e^{i\phi_1} + \epsilon\sqrt{n_2}e^{i\phi_2}\right]. \tag{L66}$$

$$\dot{n}_1 = 2\frac{\epsilon n}{\hbar}\sin\left(\phi_2 - \phi_1\right) = -\dot{n}_2 = \frac{j}{2e} \tag{L67a}$$

$$\dot{\phi}_2 - \dot{\phi}_1 = \frac{1}{\hbar}(\mathcal{E}_1 - \mathcal{E}_2) = 2e(V_2 - V_1)/\hbar. \tag{L67b}$$

$$\vec{j} = \vec{j}_0 \sin(\phi_2 - \phi_1 + \frac{2e}{\hbar c}\int_1^2 d\vec{s}\cdot\vec{A}) \tag{L68a}$$

$$\frac{-1}{\hbar}(\mathcal{E}_2 - \mathcal{E}_1) = 2eV/\hbar = \frac{\partial}{\partial t}\left(\phi_2 - \phi_1 + \frac{2e}{\hbar c}\int_1^2 d\vec{s}\cdot\vec{A}\right). \tag{L68b}$$

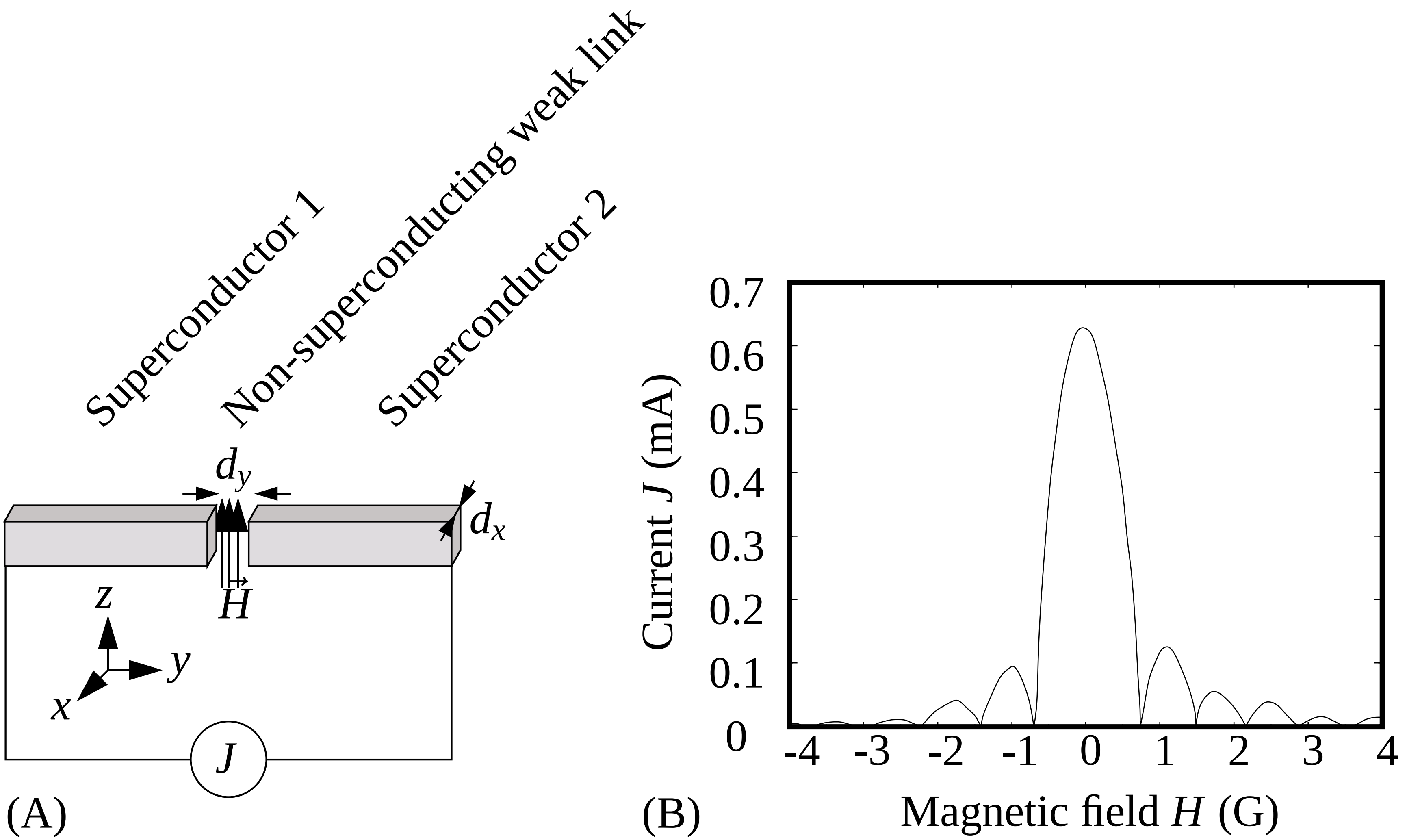


Figure 5: (A) Setting for Fraunhofer diffraction in a Josephson junction. (B) Measurement of J_c in an Sn–SnO–Sn junction at T =1.9 K. [R. C Jaklevic, 1969]

$$\frac{V}{R} + J_0 \sin\phi + C\dot{V} = J, \quad \text{(L69)}$$

$$\dot{\phi} = 2eV/\hbar, \quad \text{(L70)}$$

$$J = \frac{\dot{\phi}\hbar}{2eR} + J_0 \sin\phi + \frac{C\hbar}{2e}\ddot{\phi} \quad \text{(L71)}$$

$$\Rightarrow \frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} = -\frac{\partial}{\partial\phi}\left[-\phi J - J_0 \cos\phi\right]. \quad \text{(L72)}$$

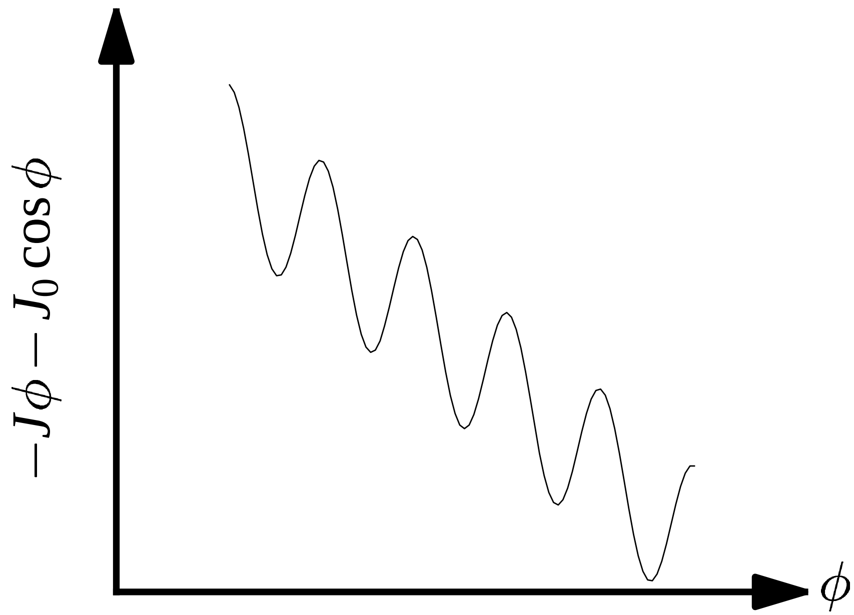


Figure 6: The washboard potential in Eq. (L72).

$$t_0 = \frac{\hbar}{2eJ_0R}, \tag{L73}$$

$$\beta\ddot{\phi} + \dot{\phi} = -\frac{\partial}{\partial\phi}\left[-\phi\frac{J}{J_0} - \cos\phi\right], \tag{L74}$$

$$\beta = \frac{J_0R^2C2e}{\hbar}. \tag{L75}$$

$$\oint d\vec{s}\cdot\vec{A} = \Phi = \int_4^1 d\vec{s}\cdot\vec{A} - \frac{\Phi_0}{4\pi}\int_1^2 d\vec{s}\cdot\vec{\nabla}\phi + \int_2^3 d\vec{s}\cdot\vec{A} - \frac{\Phi_0}{4\pi}\int_3^4 d\vec{s}\cdot\vec{\nabla}\phi \quad \text{(L76)}$$

$$\Rightarrow \Phi = \frac{\Phi_0}{4\pi}(\gamma_{23} - \gamma_{14}), \quad \text{(L77)}$$

where

$$\gamma_{14} = \phi_4 - \phi_1 + \frac{4\pi}{\Phi_0}\int_1^4 d\vec{s}\cdot\vec{A}. \quad \text{(L78)}$$

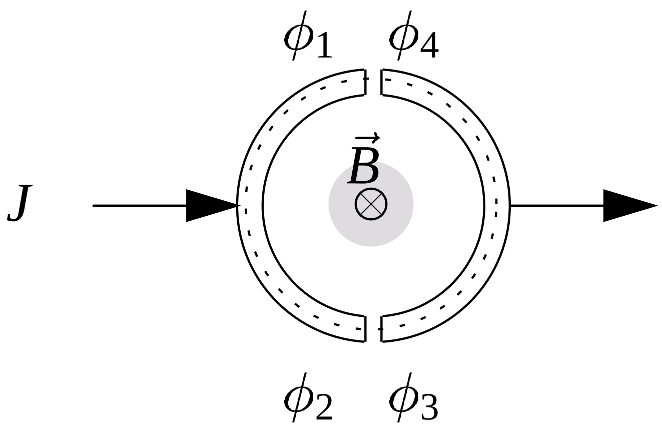


Figure 7: DC SQUID.

$$J = J_0 \sin(\gamma_{14}) + J_0 \sin(\gamma_{23}) \quad \text{(L79)}$$

$$= J_0 \Big[\sin(\gamma_{23} - 4\pi\Phi/\Phi_0) + \sin(\gamma_{23}) \Big]. \quad \text{(L80)}$$

$$\vec{j} = -\frac{|\Psi_0|^2 8\pi e\hbar}{m^\star \Phi_0}\left[\frac{\Phi_0}{4\pi}\vec{\nabla}\phi + \vec{A}\right]. \tag{L81}$$

$$L = \int d\vec{r}dt\,\mathcal{L} = \int d\vec{r}dt\left\{\frac{E^2 - B^2}{8\pi} - G(\vec{A} + \vec{\nabla}\chi, V - \dot{\chi}/c)\right\}. \tag{L82}$$

$$\vec{E} = -\vec{\nabla}V - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \text{and} \quad \vec{B} = \vec{\nabla}\times\vec{A}. \tag{L83}$$

$$\frac{\delta L}{\delta V} = 0 \Rightarrow \frac{\partial G}{\partial V} = -ne \tag{L84a}$$

$$\frac{\delta L}{\delta \vec{A}} = 0 \Rightarrow \frac{\partial G}{\partial \vec{A}} = -\frac{\vec{j}}{c}. \tag{L84b}$$

$$\frac{\delta L}{\delta \chi} = 0 \Rightarrow \vec{\nabla}\cdot\frac{\partial G}{\partial \vec{A}} - \frac{\partial}{\partial t}\frac{1}{c}\frac{\partial G}{\partial V} = 0 \tag{L85}$$

$$\Rightarrow \frac{\partial}{\partial t}[-ne] = -\vec{\nabla}\cdot\vec{j}. \tag{L86}$$

Origin of Josephson's Equations

$$\mathcal{H} = \dot{\vec{A}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} + \dot{\chi} \frac{\partial \mathcal{L}}{\partial \dot{\chi}} - \mathcal{L}. \tag{L87}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\chi}} = -\frac{ne}{c}. \tag{L88}$$

$$\frac{\partial \mathcal{H}}{\partial \chi} = \frac{\dot{n}e}{c} \tag{L89a}$$

$$\frac{\partial \mathcal{H}}{\partial [-ne/c]} = \dot{\chi}. \tag{L89b}$$

$$\dot{\chi} = -\frac{c\mu}{e} \quad \Rightarrow \dot{\phi} = -\frac{2\mu}{\hbar} = \frac{2eV}{\hbar}. \tag{L90}$$

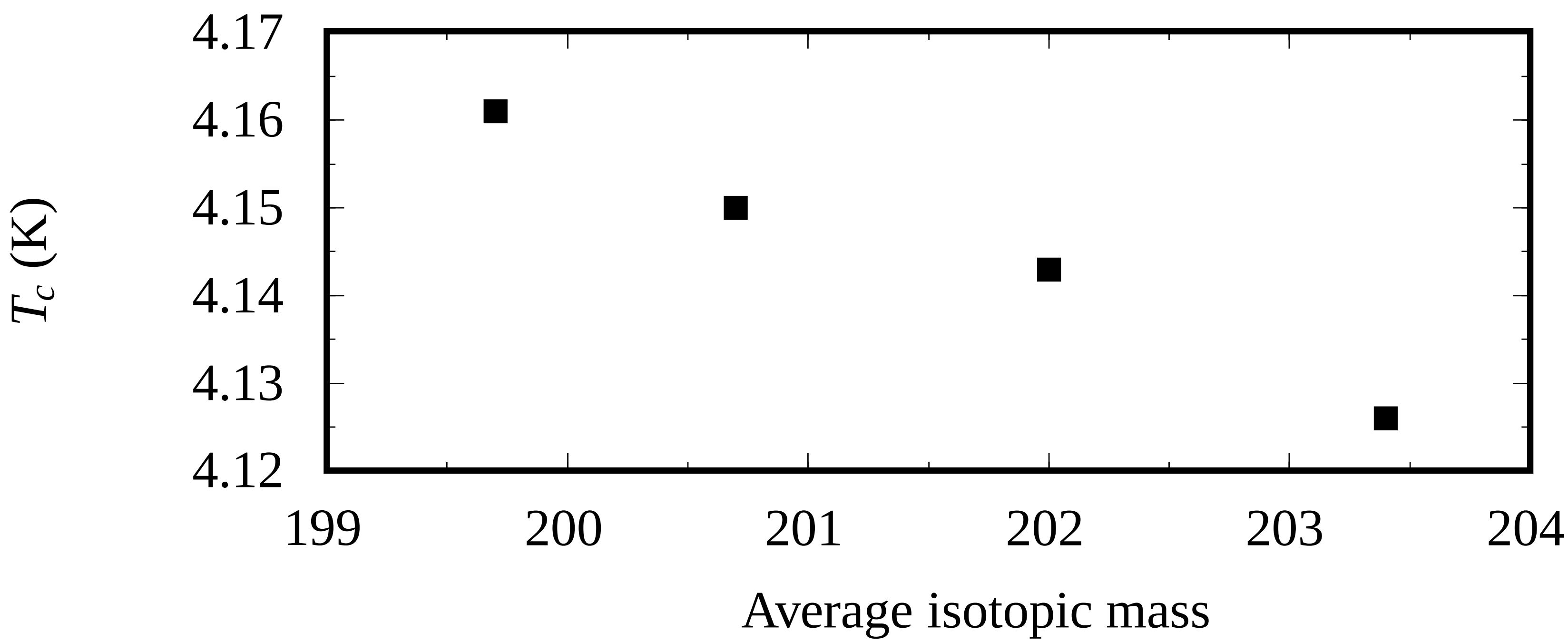


Figure 8: Superconducting transition temperature T_c versus average isotopic mass in four samples of mercury. [Reynolds et al. (1950).]

$$\sigma_{\text{el}} = \frac{i\omega\chi_{\text{c}}}{q^2}. \tag{L91}$$

$$\chi_{\text{c}} = -\frac{me^2}{\pi^2\hbar^2}\frac{(4k_F^2 - q^2)\log\left(\frac{q+2k_F}{2k_F - q}\right) + 4k_F q}{8q} \tag{L92}$$

$$\chi_{\text{c}} = -\frac{me^2 k_F}{\pi^2\hbar^2} \equiv -\frac{\kappa_{\text{c}}^2}{4\pi}. \tag{L93}$$

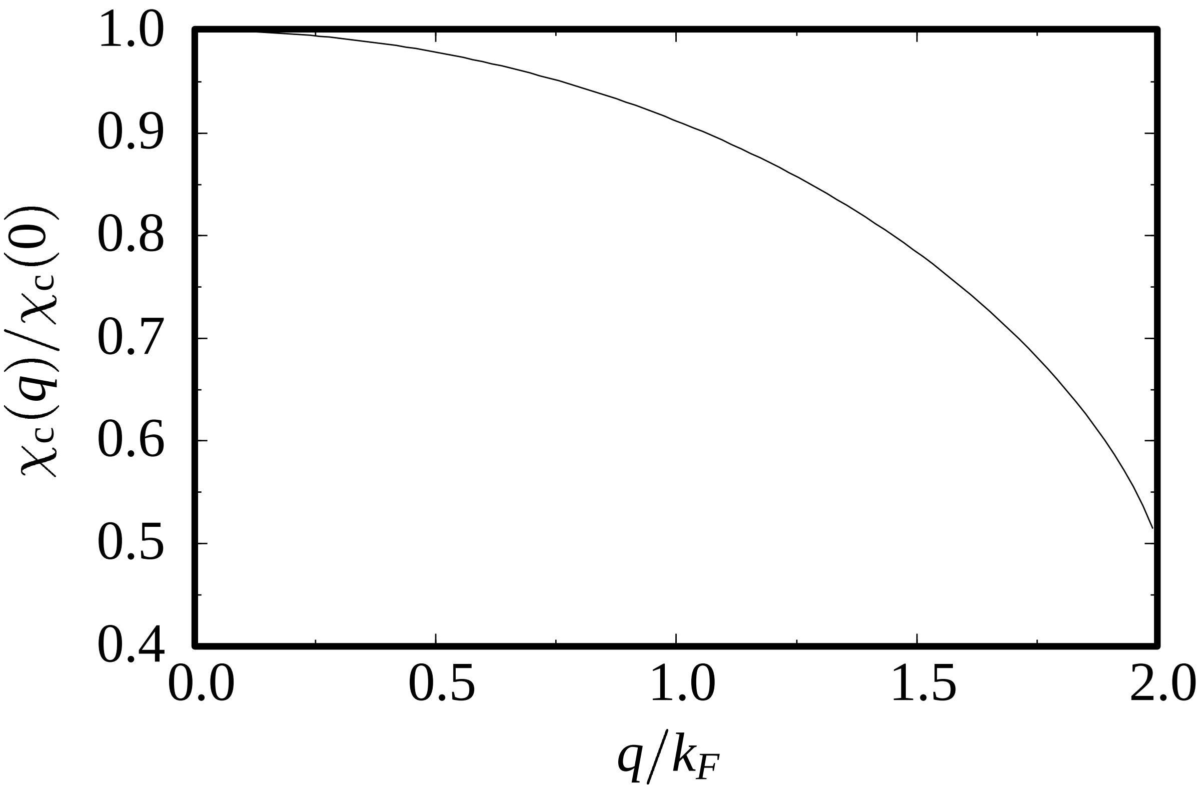


Figure 9: Charge susceptibility χ_c.

$$\sigma_{el} = \frac{\omega \kappa_c^2}{4\pi i q^2}. \tag{L94}$$

Electron–Ion Interaction

$$\vec{u} = \frac{-e^\star \vec{E}}{M(\omega^2 - \bar{\omega}_{\vec{q}}^2)}. \tag{L95}$$

$$\vec{j}_{\text{ion}}(\vec{q},\omega) = -i\omega n e^\star \vec{u} \tag{L96}$$

$$\omega_{\text{pi}}^2 = \frac{4\pi n e^{\star 2}}{M}, \tag{L97}$$

$$\sigma_{\text{ion}} = -\frac{\omega}{4\pi i}\frac{\omega_{\text{pi}}^2}{\omega^2 - \bar{\omega}_{\vec{q}}^2}. \tag{L98}$$

$$\sigma(\vec{q},\omega) = \frac{\omega}{4\pi i}\left[\frac{\kappa_c^2}{q^2} - \frac{\omega_{\text{pi}}^2}{\omega^2 - \bar{\omega}_{\vec{q}}^2}\right] \tag{L99}$$

$$\Rightarrow \epsilon(\vec{q},\omega) = 1 + \frac{\kappa_c^2}{q^2} - \frac{\omega_{\text{pi}}^2}{\omega^2 - \bar{\omega}_{\vec{q}}^2}. \tag{L100}$$

$$\omega_{\vec{q}}^2 = \bar{\omega}_{\vec{q}}^2 + \frac{q^2 \omega_{\rm pi}^2}{q^2 + \kappa_{\rm c}^2}. \tag{L101}$$

$$\frac{1}{\epsilon(\vec{q},\omega)} = \frac{q^2}{q^2+\kappa_{\rm c}^2}\left[\frac{\omega^2 - \bar{\omega}_{\vec{q}}^2}{\omega^2 - \omega_{\vec{q}}^2}\right]. \tag{L102}$$

$$|\psi_1 e^{i\vec{k}_1\cdot\vec{r} - \mathcal{E}_1 t/\hbar} + \psi_2 e^{i\vec{k}_2\cdot\vec{r} - \mathcal{E}_2 t/\hbar}|^2 \propto \text{const.} + \cos[(\vec{k}_1 - \vec{k}_2)\cdot\vec{r} - (\mathcal{E}_1 - \mathcal{E}_2)t/\hbar]. \tag{L103}$$

$$U_{\rm eff} \quad = \quad \frac{4\pi e^2}{\epsilon(\vec{q},\omega)q^2} = \frac{4\pi e^2}{q^2+\kappa_{\rm c}^2}\left[1 + \frac{\omega_{\vec{q}}^2 - \bar{\omega}_{\vec{q}}^2}{\omega^2 - \omega_{\vec{q}}^2}\right] \tag{L104a}$$

with

$$\vec{q} \quad = \quad \vec{k}_1 - \vec{k}_2 \quad \text{and} \quad \hbar\omega = \mathcal{E}_1 - \mathcal{E}_2. \tag{L104b}$$

Formal Derivation

$$\hat{U}_{\text{el}-\text{phon}} = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\substack{\vec{q}''\vec{k} \\ \sigma}} [C^*_{\vec{k}} \hat{c}^\dagger_{\vec{q}''-\vec{k},\sigma} \hat{c}_{\vec{q}''\sigma} \hat{a}^\dagger_{\vec{k}} + C_{\vec{k}} \hat{c}^\dagger_{\vec{q}''+\vec{k},\sigma} \hat{c}_{\vec{q}'',\sigma} \hat{a}_{\vec{k}}] \quad \text{(L105)}$$

$$= \frac{1}{\sqrt{\mathcal{V}}} \sum_{\substack{\vec{q}'\vec{q} \\ \sigma}} C_{\vec{q}} [\hat{c}^\dagger_{\vec{q}'+\vec{q},\sigma} \hat{c}_{\vec{q}',\sigma} \hat{a}^\dagger_{-\vec{q}} + \hat{c}^\dagger_{\vec{q}'+\vec{q},\sigma} \hat{c}_{\vec{q}',\sigma} \hat{a}_{\vec{q}}]. \quad \text{(L106)}$$

$$\epsilon_{\text{el}}(\vec{q},\omega) = \frac{q^2+\kappa_{\text{c}}^2}{q^2}, \quad \text{(L107)}$$

$$\hat{\mathcal{H}}_{\substack{\text{screened} \\ \text{Coulomb}}} = \frac{1}{\mathcal{V}} \sum_{\substack{\vec{q},\vec{k},\vec{k}' \\ \sigma,\sigma'}} \frac{1}{2} \frac{4\pi e^2}{q^2+\kappa_{\text{c}}^2} \hat{c}^\dagger_{\vec{k}'-\vec{q},\sigma'} \hat{c}^\dagger_{\vec{k}+\vec{q},\sigma} \hat{c}_{\vec{k},\sigma} \hat{c}_{\vec{k}',\sigma'}. \quad \text{(L108)}$$

$$\hat{\mathcal{H}} = \sum_{\vec{q},\sigma} \epsilon_{\vec{q}} \hat{c}^{\dagger}_{\vec{q}\sigma} \hat{c}_{\vec{q}\sigma} + \frac{1}{\mathcal{V}} \sum_{\substack{\vec{q},\vec{k},\vec{k}' \\ \sigma,\sigma'}} \frac{1}{2} \frac{4\pi e^2}{q^2 + \kappa_c^2} \hat{c}^{\dagger}_{\vec{k}'-\vec{q},\sigma'} \hat{c}^{\dagger}_{\vec{k}+\vec{q},\sigma} \hat{c}_{\vec{k},\sigma} \hat{c}_{\vec{k}',\sigma'}$$
$$+ \sum_{\vec{k}} \hbar\omega_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{\sqrt{\mathcal{V}}} \sum_{\substack{\vec{q},\vec{q}' \\ \sigma}} \hat{c}^{\dagger}_{\vec{q}+\vec{q}',\sigma} \hat{c}_{\vec{q}',\sigma} C_{\vec{q}} \left[\hat{a}_{\vec{q}} + \hat{a}^{\dagger}_{-\vec{q}} \right]. \tag{L109}$$

$$e^{-\hat{S}} \tilde{a}_{\vec{k}} e^{\hat{S}} = \hat{a}_{\vec{k}} \tag{L110}$$

$$e^{-\hat{S}} \tilde{c}_{\vec{k}\sigma} e^{\hat{S}} = \hat{c}_{\vec{k}\sigma}. \tag{L111}$$

$$\hat{\mathcal{H}} = e^{-\hat{S}} \tilde{\mathcal{H}} e^{\hat{S}}, \tag{L112}$$

$$e^{-\hat{S}} \tilde{\mathcal{H}} e^{\hat{S}} = \tilde{\mathcal{H}} + \left[\tilde{\mathcal{H}}, \hat{S}\right] + \frac{1}{2} \left[\left[\tilde{\mathcal{H}}, \hat{S}\right], \hat{S} \right] + \dots. \tag{L113}$$

$$\hat{\mathcal{H}} \approx \tilde{\mathcal{H}}_0 + \left[\tilde{\mathcal{H}}_0, \hat{S}\right] + \tilde{\mathcal{H}}_1 + \frac{1}{2}\left[\left[\tilde{\mathcal{H}}_0, \hat{S}\right], \hat{S}\right] + \left[\tilde{\mathcal{H}}_1, \hat{S}\right] \qquad \text{(L114)}$$

$$= \tilde{\mathcal{H}}_0 + \frac{1}{2}\left[\tilde{\mathcal{H}}_1, \hat{S}\right], \qquad \text{(L115)}$$

just so long as

$$0 = \left[\tilde{\mathcal{H}}_0, \hat{S}\right] + \tilde{\mathcal{H}}_1. \qquad \text{(L116)}$$

$$\hat{\mathcal{H}} = \frac{1}{2\mathcal{V}} \sum_{\substack{\vec{q}\vec{k}\vec{k}' \\ \sigma\sigma'}} \left[\frac{2|C_{\vec{q}}|^2\hbar\omega_{\vec{q}}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})^2 - \hbar^2\omega_{\vec{q}}^2} + \frac{4\pi e^2}{q^2 + \kappa_c^2}\right] \tilde{c}^{\dagger}_{\vec{k}'-\vec{q}\sigma'} \tilde{c}^{\dagger}_{\vec{k}+\vec{q}\sigma} \tilde{c}_{\vec{k}\sigma} \tilde{c}_{\vec{k}'\sigma'}. \qquad \text{(L117)}$$

Cooper Problem

$$|G\rangle = \prod_{k<k_F} \hat{c}^{\dagger}_{\vec{k}} |\varnothing\rangle. \tag{L118}$$

$$[\frac{-\hbar^2\nabla_1^2}{2m} + \frac{-\hbar^2\nabla_2^2}{2m} + U(\vec{r}_1 - \vec{r}_2)]\Psi(\vec{r}_1,\vec{r}_2) = \mathcal{E}\Psi(\vec{r}_1,\vec{r}_2), \tag{L119}$$

$$\Psi(\vec{r}_1,\vec{r}_2) = \sum_{k'>k_F} \Psi_{\vec{k}'} e^{-i\vec{k}'\cdot(\vec{r}_1-\vec{r}_2)}. \tag{L120}$$

$$(2\epsilon_{\vec{k}} - \mathcal{E})\Psi_{\vec{k}} + \sum_{k'>k_F} U_{\vec{k}\vec{k}'}\Psi_{\vec{k}'} = 0. \tag{L121}$$

$$U_{\vec{k}\vec{k}'} \quad = \quad -\frac{U_0}{\mathcal{V}}\theta(\hbar\omega - |\mathcal{E}_F - \epsilon_{\vec{k}}|)\theta(\hbar\omega - |\mathcal{E}_F - \epsilon_{\vec{k}'}|). \tag{L122}$$

Cooper Problem

$$(2\epsilon_{\vec{k}} - \mathcal{E})\Psi_{\vec{k}} = \frac{U_0}{\mathcal{V}} \sum_{k'>k_F}^{k_{\max}} \Psi_{\vec{k}'}. \tag{L123}$$

$$\mathcal{E} = 2\epsilon_{\vec{k}_a}, \tag{L124}$$

$$\Psi_{\vec{k}_a} = -\Psi_{\vec{k}_b} = \frac{1}{\sqrt{2}}. \tag{L125}$$

$$\sum_{\vec{k}>k_F}^{k_{\max}} \Psi_{\vec{k}} = \sum_{\vec{k}>k_F}^{k_{\max}} \frac{U_0}{\mathcal{V}} \frac{1}{(2\epsilon_{\vec{k}} - \mathcal{E})} \sum_{\vec{k}'>k_F}^{k_{\max}} \Psi_{\vec{k}'}. \tag{L126}$$

$$\Rightarrow 1 = \sum_{\vec{k}}^{k_{\max}} \frac{U_0}{(2\epsilon_{\vec{k}} - \mathcal{E})\mathcal{V}} \tag{L127}$$

$$\approx \int_{\mathcal{E}_F}^{\mathcal{E}_F+\hbar\omega} d\epsilon \, \frac{D(\mathcal{E}_F)}{2} \frac{U_0}{2\epsilon - \mathcal{E}} \tag{L128}$$

$$\Rightarrow 1 \quad = \quad \frac{1}{4} D(\mathcal{E}_F) U_0 \ln\left(\frac{2\mathcal{E}_F + 2\hbar\omega - \mathcal{E}}{2\mathcal{E}_F - \mathcal{E}}\right). \tag{L129}$$

$$\mathcal{E} = 2\mathcal{E}_F - (2\mathcal{E}_{\text{max}} - 2\mathcal{E}_F) \exp\left[-\frac{4}{D(\mathcal{E}_F)U_0}\right]. \tag{L130}$$

$$\Psi_{\vec{k}} = \frac{U_0}{(2\epsilon_{\vec{k}} - \mathcal{E})\mathcal{V}} \sum_{\vec{k}' > k_F}^{k_{\text{max}}} \Psi_{\vec{k}'}. \tag{L131}$$

$$|\Psi\rangle = \sum_{\vec{k}} \Psi_{\vec{k}} \hat{c}^\dagger_{\vec{k}\uparrow} \hat{c}^\dagger_{-\vec{k}\downarrow} |G\rangle. \tag{L132}$$

$$\mathcal{H} = \sum_{\vec{k}\sigma} \epsilon_{\vec{k}} \hat{c}^\dagger_{\vec{k}\sigma} \hat{c}_{\vec{k}\sigma} + \frac{1}{2\mathcal{V}} \sum_{\substack{\vec{q}\vec{k}\vec{k}' \\ \sigma\sigma'}} \left[\frac{2|C_{\vec{q}}|^2 \hbar\omega_{\vec{q}}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})^2 - \hbar^2\omega^2_{\vec{q}}} + \frac{4\pi e^2}{q^2 + \kappa_c^2}\right] \hat{c}^\dagger_{\vec{k}'-\vec{q}\sigma'} \hat{c}^\dagger_{\vec{k}+\vec{q}\sigma} \hat{c}_{\vec{k}\sigma} \hat{c}_{\vec{k}'\sigma'} \tag{L133}$$

$$\langle\hat{K}\rangle = \sum_{\vec{k}_1\vec{k}_0\vec{q}\sigma} \Psi^*_{\vec{k}_1}\Psi_{\vec{k}_0}\langle G|\hat{c}_{-\vec{k}_1\downarrow}\hat{c}_{\vec{k}_1\uparrow}\,\epsilon_{\vec{q}}\hat{c}^\dagger_{\vec{q}\sigma}\hat{c}_{\vec{q}\sigma}\,\hat{c}^\dagger_{\vec{k}_0\uparrow}\hat{c}^\dagger_{-\vec{k}_0\downarrow}|G\rangle. \tag{L134}$$

$$\vec{k}_0 = \vec{k}_1. \tag{L135}$$

$$(2\sum_{q<k_F}\epsilon_{\vec{q}})(\sum_{k_0>k_F}|\Psi_{\vec{k}_0}|^2). \tag{L136}$$

$$\sigma = \uparrow \quad \text{and} \quad \vec{q} = \vec{k}_0 \quad \text{or} \quad \sigma = \downarrow \quad \text{and} \quad \vec{q} = -\vec{k}_0. \tag{L137}$$

$$\langle\hat{K}\rangle = (2\sum_{q<k_F}\epsilon_{\vec{q}})(\sum_{k_0>k_F}|\Psi_{\vec{k}_0}|^2) + \sum_{k_0>k_F} 2|\Psi_{\vec{k}_0}|^2\epsilon_{\vec{k}_0}. \tag{L138}$$

$$\hat{\mathcal{H}} = \sum_{\substack{\vec{q}\vec{k}\vec{k}' \\ \sigma\sigma'}} U^{\text{eff}}_{\vec{k}\vec{k}'}\hat{c}^\dagger_{-\vec{k}'+\vec{q},\sigma'}\hat{c}^\dagger_{\vec{k}',\sigma}\hat{c}_{\vec{k},\sigma}\hat{c}_{-\vec{k}+\vec{q},\sigma'}, \tag{L139}$$

$$U^{\text{eff}}_{\vec{k}\vec{k}'} = \frac{1}{2\mathcal{V}}\left[\frac{2|C_{\vec{k}-\vec{k}'}|^2\hbar\omega_{\vec{k}-\vec{k}'}}{(\epsilon_{\vec{k}}-\epsilon_{\vec{k}'})^2-\hbar^2\omega^2_{\vec{k}-\vec{k}'}}+\frac{4\pi e^2}{|\vec{k}-\vec{k}'|^2+\kappa_c^2}\right]. \tag{L140}$$

$$2\sum_{kk'>k_F} U^{\text{eff}}_{\vec{k},\vec{k}'}\Psi^*_{\vec{k}'}\Psi_{\vec{k}}, \tag{L141}$$

$$2\sum_{k>k_F}\epsilon^{\text{eff}}_{\vec{k}}|\Psi_{\vec{k}}|^2+2\sum_{kk'>k_F}\Psi^*_{\vec{k}'}\Psi_{\vec{k}}U^{\text{eff}}_{\vec{k}\vec{k}'}. \tag{L142}$$

$$2\epsilon^{\text{eff}}_{\vec{k}}\Psi_{\vec{k}}+2\sum_{k'>k_F}U^{\text{eff}}_{\vec{k}\vec{k}'}\Psi_{\vec{k}'}=\mathcal{E}\Psi_{\vec{k}}. \tag{L143}$$

$$\hat{\mathcal{H}}_{\mathrm{BCS}} = \sum_{\vec{k},\sigma} \epsilon_{\vec{k}} \hat{c}^{\dagger}_{\vec{k}\sigma} \hat{c}_{\vec{k}\sigma} + \sum_{\vec{k}\vec{k}'} U_{\vec{k}\vec{k}'} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow} \hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow}. \quad \text{(L144)}$$

$$|\Phi_N\rangle = \left[\sum_{\vec{k}} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow} g_{\vec{k}} \right]^N |\varnothing\rangle, \quad \text{(L145)}$$

$$|\Phi\rangle \equiv \sum_N \frac{1}{N!} |\Phi_N\rangle \quad \text{(L146)}$$

$$= \sum_N \frac{1}{N!} \left[\sum_{\vec{k}} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow} g_{\vec{k}} \right]^N |\varnothing\rangle. \quad \text{(L147)}$$

$$|\Phi\rangle = \exp[\sum_{\vec{k}} g_{\vec{k}} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow}] |\varnothing\rangle. \quad \text{(L148)}$$

$$|\Phi\rangle = \prod_{\vec{k}} \left[1 + g_{\vec{k}} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow}\right] |\varnothing\rangle \equiv \hat{\Phi}|\varnothing\rangle. \tag{L149}$$

$$\langle\Phi|\Phi\rangle = \prod_{\vec{k}} \left(1 + |g_{\vec{k}}|^2\right) = \mathcal{N}^2. \tag{L150}$$

$$b_{\vec{k}} = \frac{1}{\mathcal{N}^2} \langle\Phi|\hat{c}_{-\vec{k}\downarrow} \hat{c}_{\vec{k}\uparrow}|\Phi\rangle = \frac{g_{\vec{k}}}{1 + |g_{\vec{k}}|^2}, \tag{L151}$$

$$\frac{1}{\mathcal{N}^2} \langle\Phi|\hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow} \hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow}|\Phi\rangle = b^*_{\vec{k}} b_{\vec{k}'}. \tag{L152}$$

$$\left[\sum_{\sigma} \hat{n}_{\vec{k}\sigma}, \hat{\Phi}\right] = \left[g_{\vec{k}} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow} + g_{-\vec{k}} \hat{c}^{\dagger}_{-\vec{k}\uparrow} \hat{c}^{\dagger}_{\vec{k}\downarrow}\right] \hat{\Phi}. \tag{L153}$$

$$\frac{1}{\mathcal{N}^2} \langle\Phi| \sum_{\sigma} \hat{n}_{\vec{k}\sigma} |\Phi\rangle = \frac{1}{\mathcal{N}^2} \langle\Phi| \left(g_{\vec{k}} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{-\vec{k}\downarrow} + g_{-\vec{k}} \hat{c}^{\dagger}_{-\vec{k}\uparrow} \hat{c}^{\dagger}_{\vec{k}\downarrow}\right) |\Phi\rangle \tag{L154}$$

$$\Rightarrow \sum_{\sigma} n_{\vec{k}\sigma} = g_{\vec{k}} b^*_{\vec{k}} + g_{-\vec{k}} b^*_{-\vec{k}}. \tag{L155}$$

$$\langle\Phi|\hat{\mathcal{H}}_{\rm BCS}-\mu N|\Phi\rangle=\sum_{\vec{k}}2\left(\epsilon_{\vec{k}}-\mu\right)g_{\vec{k}}b^*_{\vec{k}}+\sum_{\vec{k}\vec{k}'}U_{\vec{k}\vec{k}'}b^*_{\vec{k}}b_{\vec{k}'}. \quad \text{(L156)}$$

$$\frac{\partial b^*_{\vec{k}}}{\partial g^*_{\vec{k}}}=\frac{1}{\left(1+|g_{\vec{k}}|^2\right)^2};\ \frac{\partial b_{\vec{k}}}{\partial g^*_{\vec{k}}}=-\frac{g^2_{\vec{k}}}{\left(1+|g_{\vec{k}}|^2\right)^2}, \quad \text{(L157)}$$

$$\frac{2(\epsilon_{\vec{q}}-\mu)g_{\vec{q}}}{(1+|g_{\vec{q}}|^2)^2}+\sum_{\vec{k}\vec{k}'}\frac{U_{\vec{k}\vec{k}'}}{(1+|g_{\vec{q}}|^2)^2}\left[b_{\vec{k}'}\delta_{\vec{k}\vec{q}}-b^*_{\vec{k}}g^2_{\vec{q}}\delta_{\vec{q}\vec{k}'}\right]=0. \quad \text{(L158)}$$

$$\Delta_{\vec{k}}=-\sum_{\vec{k}'}U_{\vec{k}\vec{k}'}b_{\vec{k}'}, \quad \text{(L159)}$$

$$0 = 2\left(\epsilon_{\vec{q}}-\mu\right)g_{\vec{q}}-\Delta_{\vec{q}}+g^2_{\vec{q}}\Delta^*_{\vec{q}} \quad \text{(L160)}$$

$$\Rightarrow g_{\vec{k}} = \frac{\mathcal{E}_{\vec{k}}-(\epsilon_{\vec{k}}-\mu)}{\Delta^*_{\vec{k}}}, \quad \text{(L161)}$$

with

$$\mathcal{E}_{\vec{k}} = \sqrt{(\epsilon_{\vec{k}}-\mu)^2+|\Delta_{\vec{k}}|^2}. \quad \text{(L162)}$$

$$b_{\vec{k}} = \frac{g_{\vec{k}}}{1+|g_{\vec{k}}|^2} = \frac{\Delta_{\vec{k}}}{2\mathcal{E}_{\vec{k}}}. \tag{L163}$$

$$N = 2\sum_{\vec{k}} \theta(\mathcal{E}_F - \epsilon_{\vec{k}}) = \int_0^{\mathcal{E}_F} d\epsilon D(\epsilon), \tag{L164}$$

$$N = \sum_{\vec{k}\sigma} g^*_{\vec{k}} b_{\vec{k}} = \sum_{\vec{k}\sigma} \frac{1}{2}\left[1 - \frac{\epsilon_{\vec{k}} - \mu}{\mathcal{E}_{\vec{k}}}\right] \tag{L165}$$

$$= \sum_{\vec{k}\sigma} \frac{1}{2}\left[1 - \frac{\epsilon_{\vec{k}} - \mu}{\sqrt{(\epsilon_{\vec{k}} - \mu)^2 + |\Delta|^2}}\right] \tag{L166}$$

$$= \int d\epsilon \frac{D(\epsilon)}{2}\left[1 - \frac{\epsilon - \mu}{\sqrt{(\epsilon - \mu)^2 + |\Delta|^2}}\right] \tag{L167}$$

$$= \int d\epsilon \left[\int^{\epsilon} d\epsilon' D(\epsilon')\right] \frac{1}{2}\frac{\partial}{\partial \epsilon}\frac{\epsilon - \mu}{\sqrt{(\epsilon - \mu)^2 + |\Delta|^2}} \tag{L168}$$

$$= \int d\epsilon \left[\int^{\mu} d\epsilon' D(\epsilon')\right] \frac{1}{2}\frac{\partial}{\partial \epsilon}\frac{\epsilon - \mu}{\sqrt{(\epsilon - \mu)^2 + |\Delta|^2}} + \mathcal{O}(\Delta/\mathcal{E}_F)^2 \tag{L169}$$

$$= \left[\int^{\mu} d\epsilon' D(\epsilon')\right] = N + D(\mathcal{E}_F)(\mu - \mathcal{E}_F) \quad \text{(L170)}$$

$$\Rightarrow \mu = \mathcal{E}_F. \quad \text{(L171)}$$

$$\Delta_{\vec{k}} = -\sum_{\vec{k}'} U_{\vec{k}\vec{k}'} \frac{\Delta_{\vec{k}'}}{2\mathcal{E}_{\vec{k}'}}. \quad \text{(L172)}$$

$$\Delta_{\vec{k}} = \theta(\hbar\omega - |\epsilon_{\vec{k}} - \mathcal{E}_F|) \frac{U_0}{\mathcal{V}} \sum_{\vec{k}'} \frac{\Delta_{\vec{k}'}}{2\sqrt{(\epsilon_{\vec{k}'} - \mu)^2 + |\Delta_{\vec{k}'}|^2}}. \quad \text{(L173)}$$

$$\Delta_{\vec{k}} = \Delta\theta(\hbar\omega - |\epsilon_{\vec{k}} - \mathcal{E}_F|). \quad \text{(L174)}$$

$$1 = \sum_{\vec{k}} \frac{1}{\mathcal{V}} \theta(\hbar\omega - |\epsilon_{\vec{k}} - \mathcal{E}_F|) \frac{U_0}{2\sqrt{(\epsilon_{\vec{k}} - \mu)^2 + |\Delta|^2}}. \quad \text{(L175)}$$

$$= \int_{\mathcal{E}_F - \hbar\omega}^{\mathcal{E}_F + \hbar\omega} d\epsilon \frac{D(\epsilon)}{2} \frac{U_0}{2\sqrt{(\epsilon - \mathcal{E}_F)^2 + |\Delta|^2}} \quad \text{(L176)}$$

$$= U_0 \int_0^{\hbar\omega/\Delta} d\zeta \, \frac{D(\mathcal{E}_F)}{2\sqrt{\zeta^2+1}} \quad \text{(L177)}$$

$$= \frac{U_0 D(\mathcal{E}_F)}{2} \sinh^{-1} \frac{\hbar\omega}{\Delta} \quad \text{(L178)}$$

$$\Rightarrow \Delta = 2\hbar\omega \exp\left[-\frac{2}{D(\mathcal{E}_F)U_0}\right]. \quad \text{(L179)}$$

$$Z_{\rm gr} = {\rm Tr} e^{-\beta[\hat{\mathcal{H}}_{\rm BCS} - \mu\hat{N}]}, \tag{L180}$$

$$\hat{c}_{-\vec{k}\downarrow}\hat{c}_{\vec{k}\uparrow} = b_{\vec{k}} + \left(\hat{c}_{-\vec{k}\downarrow}\hat{c}_{\vec{k}\uparrow} - b_{\vec{k}}\right), \tag{L181}$$

$$Z_{\rm gr} = {\rm Tr} e^{-\beta[\hat{\mathcal{H}}_{\rm eff} - \mu N]}, \tag{L182}$$

where

$$\begin{aligned}
&\hat{\mathcal{H}}_{\rm eff} - \mu N \\
&= \sum_{\vec{k}\sigma} \hat{n}_{\vec{k}\sigma}(\epsilon_{\vec{k}} - \mu) + \sum_{\vec{k}\vec{k}'} b_{\vec{k}'} U_{\vec{k}\vec{k}'} \hat{c}^\dagger_{\vec{k}\uparrow} \hat{c}^\dagger_{-\vec{k}\downarrow} + b^*_{\vec{k}} U^*_{\vec{k}'\vec{k}} \hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow} - b^*_{\vec{k}} b_{\vec{k}'} U_{\vec{k}\vec{k}'} && \text{(L183)} \\
&\equiv \sum_{\vec{k}\sigma} \hat{n}_{\vec{k}\sigma}(\epsilon_{\vec{k}} - \mu) - \sum_{\vec{k}} [\Delta_{\vec{k}} \hat{c}^\dagger_{\vec{k}\uparrow} \hat{c}^\dagger_{-\vec{k}\downarrow} + \Delta^*_{\vec{k}} \hat{c}_{-\vec{k}\downarrow} \hat{c}_{\vec{k}\uparrow}] - \sum_{\vec{k}\vec{k}'} b^*_{\vec{k}} b_{\vec{k}'} U_{\vec{k}\vec{k}'}. && \text{(L184)}
\end{aligned}$$

$$\begin{aligned}
\hat{c}_{\vec{k}\uparrow} &= u_{\vec{k}} \hat{\gamma}_{\vec{k}\uparrow} + v^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\downarrow} && \text{(L185a)} \\
\hat{c}^\dagger_{-\vec{k}\downarrow} &= -v_{\vec{k}} \hat{\gamma}_{\vec{k}\uparrow} + u^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\downarrow}. && \text{(L185b)}
\end{aligned}$$

$$|u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1. \tag{L186}$$

$$\hat{\mathcal{H}}_{\text{eff}} - \mu N = \sum_{\vec{k}} \left[\begin{array}{ll} (\epsilon_{\vec{k}} - \mu) & \left\{ \begin{array}{l} [\quad u^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\uparrow} + v_{\vec{k}} \hat{\gamma}_{\vec{k}\downarrow}] [\quad u_{\vec{k}} \hat{\gamma}_{\vec{k}\uparrow} + v^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\downarrow}] \\ + [- \quad v_{\vec{k}} \hat{\gamma}_{\vec{k}\uparrow} + u^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\downarrow}] [- \quad v^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\uparrow} + u_{\vec{k}} \hat{\gamma}_{\vec{k}\downarrow}] \end{array} \right\} \\ - & \Delta_{\vec{k}} [u^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\uparrow} + v_{\vec{k}} \hat{\gamma}_{\vec{k}\downarrow}] [- v_{\vec{k}} \hat{\gamma}_{\vec{k}\uparrow} + u^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\downarrow}] \\ - & \Delta^*_{\vec{k}} [- v^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\uparrow} + u_{\vec{k}} \hat{\gamma}_{\vec{k}\downarrow}] [u_{\vec{k}} \hat{\gamma}_{\vec{k}\uparrow} + v^*_{\vec{k}} \hat{\gamma}^\dagger_{\vec{k}\downarrow}] \\ - & \sum_{\vec{k}'} b^*_{\vec{k}} b_{\vec{k}'} U_{\vec{k}\vec{k}'}. \end{array} \right] \tag{L187}$$

$$2u_{\vec{k}} v_{\vec{k}} (\epsilon_{\vec{k}} - \mu) + \Delta_{\vec{k}} v^2_{\vec{k}} - \Delta^*_{\vec{k}} u^2_{\vec{k}} = 0. \tag{L188}$$

$$0 = 2\sqrt{1 - |v_{\vec{k}}|^2}\, v_{\vec{k}} (\epsilon_{\vec{k}} - \mu) + \Delta_{\vec{k}} v^2_{\vec{k}} - \Delta^*_{\vec{k}} (1 - |v_{\vec{k}}|^2). \tag{L189}$$

$$v_{\vec{k}} = \frac{g^*_{\vec{k}}}{\sqrt{1+|g_{\vec{k}}|^2}}, \tag{L190}$$

$$0 = 2(\epsilon_{\vec{k}} - \mu)g_{\vec{k}} - \Delta_{\vec{k}} + g^2_{\vec{k}}\Delta^*_{\vec{k}}. \tag{L191}$$

$$g_{\vec{k}} = \frac{\mathcal{E}_{\vec{k}} - (\epsilon_{\vec{k}} - \mu)}{\Delta^*_{\vec{k}}}. \tag{L192}$$

$$|v_{\vec{k}}|^2 = \frac{\mathcal{E}_{\vec{k}} - (\epsilon_{\vec{k}} - \mu)}{2\mathcal{E}_{\vec{k}}}, \quad |u_{\vec{k}}|^2 = \frac{\mathcal{E}_{\vec{k}} + \epsilon_{\vec{k}} - \mu}{2\mathcal{E}_{\vec{k}}}, \quad v_{\vec{k}}u^*_{\vec{k}} = \frac{\Delta^*_{\vec{k}}}{2\mathcal{E}_{\vec{k}}}. \tag{L193}$$

$$\hat{\mathcal{H}}_{\text{eff}} - \mu N = \sum_{\vec{k}} \mathcal{E}_{\vec{k}} \left[\hat{\gamma}^\dagger_{\vec{k}\uparrow}\hat{\gamma}_{\vec{k}\uparrow} + \hat{\gamma}^\dagger_{\vec{k}\downarrow}\hat{\gamma}_{\vec{k}\downarrow}\right] + \sum_{\vec{k}} \left[\epsilon_{\vec{k}} - \mu - \mathcal{E}_{\vec{k}} - \sum_{\vec{k}'} b^*_{\vec{k}} b_{\vec{k}'} U_{\vec{k}\vec{k}'}\right]. \tag{L194}$$

$$b^*_{-\vec{k}} = b^*_{\vec{k}} = \left\langle \hat{c}^\dagger_{\vec{k}\uparrow}\hat{c}^\dagger_{-\vec{k}\downarrow}\right\rangle = \left\langle v_{\vec{k}}u^*_{\vec{k}}\left(\hat{\gamma}_{\vec{k}\downarrow}\hat{\gamma}^\dagger_{\vec{k}\downarrow} - \hat{\gamma}^\dagger_{\vec{k}\uparrow}\hat{\gamma}_{\vec{k}\uparrow}\right)\right\rangle + \text{terms with } \gamma\gamma \text{ or } \gamma^\dagger\gamma^\dagger. \tag{L195}$$

$$b^*_{\vec{k}} = v_{\vec{k}} u^*_{\vec{k}} \left(1 - 2 f_{\vec{k}}\right), \tag{L196}$$

$$f_{\vec{k}} = \frac{1}{e^{\beta \mathcal{E}_{\vec{k}}} + 1} \tag{L197}$$

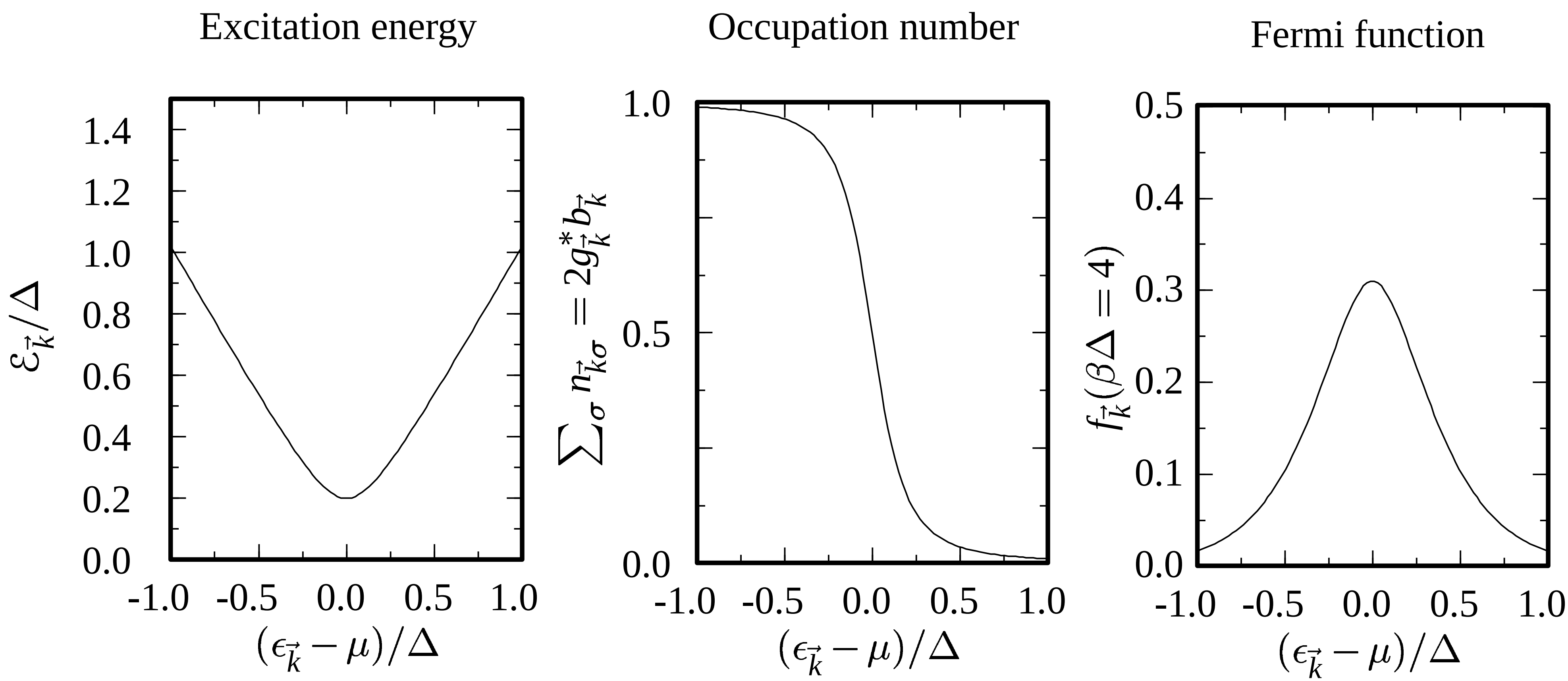


Figure 10: Sketches of the excitation energy, occupation number, and Fermi function for the BCS theory of superconductivity.

$$b_{\vec{k}} = \frac{\Delta_{\vec{k}}}{2\mathcal{E}_{\vec{k}}}\left(1 - 2f_{\vec{k}}\right) \tag{L198}$$

$$\Rightarrow \sum_{\vec{k}'} b_{\vec{k}'} U_{\vec{k}\vec{k}'} = -\Delta_{\vec{k}} = \sum_{\vec{k}'} \frac{\Delta_{\vec{k}'}}{2\mathcal{E}_{\vec{k}'}} \left(1 - 2 f_{\vec{k}'}\right) U_{\vec{k}\vec{k}'}, \tag{L199}$$

$$\Delta = \sum_{\vec{k}} \theta(\hbar\omega - |\mathcal{E}_F - \epsilon_{\vec{k}}|) \frac{\Delta}{2|\epsilon_{\vec{k}} - \mu|} \frac{U_0}{\mathcal{V}} \left(1 - 2 f_{\vec{k}}\right) \tag{L200}$$

$$\Rightarrow 1 = \int_0^{\beta\hbar\omega} U_0 \frac{D(\mathcal{E}_F)}{2} \frac{dx}{x} \left[1 - \frac{2}{e^x + 1}\right] \tag{L201}$$

$$\approx \frac{U_0 D(\mathcal{E}_F)}{2} \left\{ \ln\beta\hbar\omega \left[1 - \frac{2}{e^{\beta\hbar\omega} + 1}\right] + 2\int_0^{\infty} dx \ln x \frac{\partial}{\partial x} \frac{1}{e^x + 1} \right\} \tag{L202}$$

$$\approx \frac{U_0 D(\mathcal{E}_F)}{2} \left\{ \ln(\beta\hbar\omega) + \ln\left(\frac{2\gamma_E}{\pi}\right) \right\}, \tag{L203}$$

$$k_B T_c = \hbar\omega \frac{2\gamma_E}{\pi} \exp\left[-\frac{2}{U_0 D(\mathcal{E}_F)}\right], \tag{L204}$$

$$\Rightarrow \frac{2\Delta(T=0)}{k_B T_c} = \frac{2\pi}{\gamma_E} = 3.53. \tag{L205}$$

Thermodynamics of Superconductors

Element	$2\Delta/k_BT$	$(C_s - C_n)/C_n$	Element	$2\Delta/k_BT$	$(C_s - C_n)/C_n$
BCS	3.53	1.43			
Al	2.5–4.2	1.3–1.6	Pb	4.0–4.4	2.7
Cd	3.2–3.4	1.3–1.4	Sn	2.8–4.0	1.6
Ga	3.5	1.4	Ta	3.5–3.7	1.6
Hg	4.0–4.6	2.4	Tl	3.6–3.9	1.5
In	3.4–3.7	1.7	V	3.4–3.5	1.5
La	1.7–3.2	1.5	Zn	3.2–3.4	1.2–1.3
Nb	3.6–3.8	1.9–2.0			

$$\hat{\mathcal{H}} = \sum_{\vec{k}\vec{k}'\sigma} \epsilon_{\vec{k}\vec{k}'} \hat{c}^{\dagger}_{\vec{k}\sigma} \hat{c}_{\vec{k}'\sigma} - \sum_{\vec{k}\vec{q}\vec{k}'} \frac{U_0}{\mathcal{V}} \hat{c}^{\dagger}_{\vec{k}\uparrow} \hat{c}^{\dagger}_{\vec{q}-\vec{k}\downarrow} \hat{c}_{\vec{q}-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow}. \quad \text{(L206)}$$

$$\Delta_{\vec{q}} = \sum_{\vec{k}'} \frac{U_0}{\mathcal{V}} \left\langle \hat{c}_{\vec{q}-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow} \right\rangle. \quad \text{(L207)}$$

$$\hat{\mathcal{H}} - \mu N = \sum_{\vec{k}\vec{k}'\sigma} [\epsilon_{\vec{k}\vec{k}'} - \mu \delta_{\vec{k}\vec{k}'}] \hat{c}^{\dagger}_{\vec{k}\sigma} \hat{c}_{\vec{k}'\sigma} - \sum_{\vec{k}\vec{q}} [\Delta^*_{\vec{q}} \hat{c}_{\vec{q}-\vec{k}\downarrow} \hat{c}_{\vec{k}\uparrow} + \Delta_{\vec{q}} c^{\dagger}_{\vec{k}\uparrow} c^{\dagger}_{\vec{q}-\vec{k}\downarrow}]. \quad \text{(L208)}$$

$$\hat{c}_{\vec{r}\sigma} = \sum_{\vec{k}} \frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{N_k}} \hat{c}_{\vec{k}\sigma}, \quad \Delta_{\vec{k}} = \frac{1}{N_k} \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}} \Delta_{\vec{r}}, \quad \epsilon_{\vec{k}\vec{k}'} = \frac{1}{N_k} \sum_{\vec{r}\vec{r}'} e^{i\vec{k}\cdot\vec{r} - i\vec{k}'\cdot\vec{r}'} \epsilon_{\vec{r}\vec{r}'}. \quad \text{(L209)}$$

$$\hat{\mathcal{H}} = \sum_{\vec{r}\vec{r}'\sigma} [\epsilon_{\vec{r}\vec{r}'} - \mu \delta_{\vec{r}\vec{r}'}] \hat{c}^{\dagger}_{\vec{r}\sigma} \hat{c}_{\vec{r}'\sigma} - \sum_{\vec{r}} [\Delta^*_{\vec{r}} \hat{c}_{\vec{r}\downarrow} \hat{c}_{\vec{r}\uparrow} + \Delta_{\vec{r}} \hat{c}^{\dagger}_{\vec{r}\uparrow} \hat{c}^{\dagger}_{\vec{r}\downarrow}]. \quad \text{(L210)}$$

$$\hat{\mathcal{H}} = \sum_l \mathcal{E}_l \left[\hat{\gamma}^\dagger_{l\uparrow}\hat{\gamma}_{l\uparrow} + \hat{\gamma}^\dagger_{l\downarrow}\hat{\gamma}_{l\downarrow}\right]. \tag{L211}$$

$$\begin{aligned} \hat{c}_{\vec{r}\uparrow} &= \frac{1}{\sqrt{N_k}}\sum_l u_l(\vec{r})\hat{\gamma}_{l\uparrow} + v^*_l(\vec{r})\hat{\gamma}^\dagger_{l\downarrow} \\ \hat{c}_{\vec{r}\downarrow} &= \frac{1}{\sqrt{N_k}}\sum_l u_l(\vec{r})\hat{\gamma}_{l\downarrow} - v^*_l(\vec{r})\hat{\gamma}^\dagger_{l\uparrow}. \end{aligned} \tag{L212}$$

$$\begin{aligned} [\mathcal{H}_B, \hat{\gamma}_{l\sigma}] &= -\mathcal{E}_l\hat{\gamma}_{l\sigma} \\ \left[\mathcal{H}_B, \hat{\gamma}^\dagger_{l\sigma}\right] &= \mathcal{E}_l\hat{\gamma}^\dagger_{l\sigma}. \end{aligned} \tag{L213}$$

$$\left[\mathcal{H}_B, \hat{c}^\dagger_{\vec{r}\uparrow}\right] = \sum_{\vec{r}'}[\epsilon^*_{\vec{r}\vec{r}'} - \mu\delta_{\vec{r}\vec{r}'}]\hat{c}^\dagger_{\vec{r}'\uparrow} - \Delta^*_{\vec{r}}\hat{c}_{\vec{r}\downarrow} \tag{L214a}$$

$$\left[\mathcal{H}_B, \hat{c}^\dagger_{\vec{r}\downarrow}\right] = \sum_{\vec{r}'}[\epsilon^*_{\vec{r}\vec{r}'} - \mu\delta_{\vec{r}\vec{r}'}]\hat{c}^\dagger_{\vec{r}'\downarrow} + \Delta^*_{\vec{r}}\hat{c}_{\vec{r}\uparrow} \tag{L214b}$$

$$[\mathcal{H}_B, \hat{c}_{\vec{r}\uparrow}] = -\sum_{\vec{r}'}[\epsilon_{\vec{r}\vec{r}'} - \mu\delta_{\vec{r}\vec{r}'}]\hat{c}_{\vec{r}'\uparrow} + \Delta_{\vec{r}}\hat{c}^\dagger_{\vec{r}\downarrow} \tag{L214c}$$

$$[\mathcal{H}_B, \hat{c}_{\vec{r}\downarrow}] = -\sum_{\vec{r}'} [\epsilon_{\vec{r}\vec{r}'} - \mu\delta_{\vec{r}\vec{r}'}]\hat{c}_{\vec{r}'\downarrow} - \Delta_{\vec{r}}\hat{c}^{\dagger}_{\vec{r}\uparrow}. \quad \text{(L214d)}$$

$$\begin{aligned} u_l(\vec{r})\mathcal{E}_l &= \textstyle\sum_{\vec{r}'} [\epsilon_{\vec{r}\vec{r}'} - \mu\delta_{\vec{r}\vec{r}'}] u_l(\vec{r}') + v_l(\vec{r})\Delta_{\vec{r}} \\ v_l(\vec{r})\mathcal{E}_l &= -\textstyle\sum_{\vec{r}'} [\epsilon^*_{\vec{r}\vec{r}'} - \mu\delta_{\vec{r}\vec{r}'}] v_l(\vec{r}') + u_l(\vec{r})\Delta^*_{\vec{r}}. \end{aligned} \quad \text{(L215)}$$

$$\Delta_{\vec{r}} = \frac{U_0}{\mathcal{V}} N_k \langle \hat{c}_{\vec{r}\downarrow}\hat{c}_{\vec{r}\uparrow}\rangle = \sum_l \frac{U_0}{\mathcal{V}} u_l(\vec{r}) v^*_l(\vec{r}). \quad \text{(L216)}$$

$$u^{(0)}_{\vec{k}}(\vec{r}) = u_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}}, \quad v^{(0)}_{\vec{k}}(\vec{r}) = v_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}}. \quad \text{(L217)}$$

$$u_{\vec{k}}(\vec{r})\mathcal{E}_{\vec{k}} = \left\{ \frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{e\vec{A}}{c}\right)^2 - \mu \right\} u_{\vec{k}}(\vec{r}) + v_{\vec{k}}(\vec{r})\Delta_{\vec{r}} \quad \text{(L218a)}$$

$$v_{\vec{k}}(\vec{r})\mathcal{E}_{\vec{k}} = -\left\{ \frac{1}{2m}\left(i\hbar\vec{\nabla} + \frac{e\vec{A}}{c}\right)^2 - \mu \right\} v_{\vec{k}}(\vec{r}) + u_{\vec{k}}(\vec{r})\Delta^*_{\vec{r}}. \quad \text{(L218b)}$$

$$u_{\vec{k}}(\vec{r}) = u_{\vec{k}}^{(0)}(\vec{r}) + u_{\vec{k}}^{(1)}(\vec{r}) = u_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} + \sum_{\vec{k}'} e^{-i\vec{k}'\cdot\vec{r}} u_{\vec{k}}^{(1)}(\vec{k}') \quad \text{(L219a)}$$

$$v_{\vec{k}}(\vec{r}) = v_{\vec{k}}^{(0)}(\vec{r}) + v_{\vec{k}}^{(1)}(\vec{r}) = v_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} + \sum_{\vec{k}'} e^{-i\vec{k}'\cdot\vec{r}} v_{\vec{k}}^{(1)}(\vec{k}'). \quad \text{(L219b)}$$

$$\left(\mathcal{E}_{\vec{k}} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right) v_{\vec{k}}^{(1)}(\vec{r}) - \Delta^* u_{\vec{k}}^{(1)}(\vec{r}) = -\frac{ie\hbar}{2mc}\left(\vec{A}\cdot\vec{\nabla} + \vec{\nabla}\cdot\vec{A}\right) v_{\vec{k}}^{(0)}(\vec{r}) \quad \text{(L220a)}$$

$$\left(\mathcal{E}_{\vec{k}} + \frac{\hbar^2}{2m}\nabla^2 + \mu\right) u_{\vec{k}}^{(1)}(\vec{r}) - \Delta\, v_{\vec{k}}^{(1)}(\vec{r}) = -\frac{ie\hbar}{2mc}\left(\vec{A}\cdot\vec{\nabla} + \vec{\nabla}\cdot\vec{A}\right) u_{\vec{k}}^{(0)}(\vec{r}). \quad \text{(L220b)}$$

$$\left(\mathcal{E}_{\vec{k}} + \zeta_{\vec{k}'}\right) v_{\vec{k}}^{(1)}(\vec{k}') - \Delta^* u_{\vec{k}}^{(1)}(\vec{k}') = F_{\vec{k}'\vec{k}} v_{\vec{k}}^{(0)} \quad \text{(L221a)}$$

$$\left(\mathcal{E}_{\vec{k}} - \zeta_{\vec{k}'}\right) u_{\vec{k}}^{(1)}(\vec{k}') - \Delta v_{\vec{k}}^{(1)}(\vec{k}') = F_{\vec{k}'\vec{k}} u_{\vec{k}}^{(0)}, \quad \text{(L221b)}$$

$$\zeta_{\vec{k}} = \frac{\hbar^2 k^2}{2m} - \mu, \text{ so that } \mathcal{E}_{\vec{k}} = \sqrt{\zeta_{\vec{k}}^2 + |\Delta|^2}, \quad \text{(L222)}$$

$$F_{\vec{k}'\vec{k}} = -\frac{e\hbar}{2mc}\int \frac{d\vec{r}'}{\mathcal{V}}\, e^{i(\vec{k}'-\vec{k})\cdot\vec{r}'}(\vec{k}+\vec{k}')\cdot\vec{A}(\vec{r}') = F^*_{\vec{k}\vec{k}'}. \tag{L223}$$

$$v^{(1)}_{\vec{k}}(\vec{k}') = \frac{F_{\vec{k}'\vec{k}}}{\mathcal{E}^2_{\vec{k}} - \mathcal{E}^2_{\vec{k}'}}\left[(\mathcal{E}_{\vec{k}} - \zeta_{\vec{k}'})v_{\vec{k}} + \Delta^* u_{\vec{k}}\right] \tag{L224a}$$

$$u^{(1)}_{\vec{k}}(\vec{k}') = \frac{F_{\vec{k}'\vec{k}}}{\mathcal{E}^2_{\vec{k}} - \mathcal{E}^2_{\vec{k}'}}\left[(\mathcal{E}_{\vec{k}} + \zeta_{\vec{k}'})u_{\vec{k}} + \Delta v_{\vec{k}}\right]. \tag{L224b}$$

Derivation of Meissner Effect

$$\vec{j} = -2e\frac{N_k}{\mathcal{V}}\mathrm{Re}\left\langle \hat{c}^\dagger_{\vec{r}\uparrow}\left(\frac{\hat{P}}{m}+\frac{e\vec{A}}{mc}\right)\hat{c}_{\vec{r}\uparrow}\right\rangle \tag{L225}$$

$$= \frac{-e}{\mathcal{V}}\sum_{\vec{k}\vec{k}'}\left\langle \left(u^*_{\vec{k}'}(\vec{r})\hat{\gamma}^\dagger_{\vec{k}'\uparrow}+v_{\vec{k}'}(\vec{r})\hat{\gamma}_{\vec{k}\downarrow}\right)\left(\frac{\hat{P}}{m}+\frac{e\vec{A}}{mc}\right)\left(u_{\vec{k}}(\vec{r})\hat{\gamma}_{\vec{k}\uparrow}+v^*_{\vec{k}}(\vec{r})\hat{\gamma}^\dagger_{\vec{k}\downarrow}\right)\right\rangle$$

$$+\mathrm{c.c.} \tag{L226}$$

$$= \frac{-e}{\mathcal{V}}\sum_{\vec{k}}v_{\vec{k}}(\vec{r})\left[\frac{\hbar\vec{\nabla}}{im}+\frac{e\vec{A}}{mc}\right]v^*_{\vec{k}}(\vec{r})+\mathrm{c.c.}. \tag{L227}$$

$$\sum_{\vec{k}}v_{\vec{k}}v^*_{\vec{k}} = \frac{1}{2}\sum_{\vec{k}}\frac{\mathcal{E}_{\vec{k}}-\zeta_{\vec{k}}}{\mathcal{E}_{\vec{k}}} = N/2, \tag{L228}$$

$$\vec{j} = \vec{j}^1 - \frac{ne^2\vec{A}}{mc}, \tag{L229}$$

$$\vec{j}^1 = -e\frac{1}{\mathcal{V}}\sum_{\vec{k}}v_{\vec{k}}\frac{\hbar\vec{\nabla}}{im}v^*_{\vec{k}}(\vec{r})+\mathrm{c.c.} \tag{L230}$$

Derivation of Meissner Effect

$$\vec{j}^1 = -\frac{e\hbar}{\mathcal{V}m}\sum_{\vec{k}\vec{k}'} v_{\vec{k}} v_{\vec{k}}^{(1)*}(\vec{k}')\left(\vec{k}+\vec{k}'\right)e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} + v_{\vec{k}}^* v_{\vec{k}}^{(1)}(\vec{k}')\left(\vec{k}+\vec{k}'\right)e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}. \tag{L231}$$

$$\begin{aligned}\vec{j}^1 &= \frac{-e\hbar}{m\mathcal{V}}\sum_{\vec{k}\vec{k}'} \left(\vec{k}+\vec{k}'\right)e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\frac{F_{\vec{k}'\vec{k}}}{\mathcal{E}_{\vec{k}}^2-\mathcal{E}_{\vec{k}'}^2}\left[(\mathcal{E}_{\vec{k}}-\zeta_{\vec{k}'})\left(\frac{\mathcal{E}_{\vec{k}}-\zeta_{\vec{k}}}{2\mathcal{E}_{\vec{k}}}\right)+\frac{\Delta^*\Delta}{2\mathcal{E}_{\vec{k}}}\right] \\ &\quad + \left(\vec{k}+\vec{k}'\right)e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}}\frac{F^*_{\vec{k}'\vec{k}}}{\mathcal{E}_{\vec{k}}^2-\mathcal{E}_{\vec{k}'}^2}\left[(\mathcal{E}_{\vec{k}}-\zeta_{\vec{k}'})\left(\frac{\mathcal{E}_{\vec{k}}-\zeta_{\vec{k}}}{2\mathcal{E}_{\vec{k}}}\right)+\frac{\Delta\Delta^*}{2\mathcal{E}_{\vec{k}}}\right] \qquad \text{(L232)} \\ &= \frac{-e\hbar}{m\mathcal{V}}\sum_{\vec{k}\vec{k}'}\left(\vec{k}+\vec{k}'\right)e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}F_{\vec{k}'\vec{k}}L(\zeta_{\vec{k}},\zeta_{\vec{k}'}), \qquad \text{(L233)}\end{aligned}$$

$$L(\zeta_{\vec{k}},\zeta_{\vec{k}'}) = \frac{\mathcal{E}_{\vec{k}}\mathcal{E}_{\vec{k}'}-\zeta_{\vec{k}}\zeta_{\vec{k}'}-\Delta^*\Delta}{2(\mathcal{E}_{\vec{k}}+\mathcal{E}_{\vec{k}'})\mathcal{E}_{\vec{k}}\mathcal{E}_{\vec{k}'}}. \tag{L234}$$

$$\sigma_{\alpha\beta}(\zeta,\zeta',\vec{R}) = \frac{2\pi\hbar}{2\mathcal{V}^2}\left(\frac{e\hbar}{m}\right)^2 \sum_{\vec{k}\vec{k}'} \delta(\zeta_{\vec{k}}-\zeta)\delta(\zeta_{\vec{k}'}-\zeta')\,(k_\alpha+k'_\alpha)\,(k_\beta+k'_\beta)\,e^{i(\vec{k}-\vec{k}')\cdot\vec{R}}. \tag{L235}$$

$$\vec{j}^{1}_{\alpha}(\vec{r}) = \frac{1}{2\pi\hbar c}\sum_\beta \int d\vec{r}'\,d\zeta\,d\zeta'\,L(\zeta,\zeta')\sigma_{\alpha\beta}(\zeta,\zeta',\vec{r}-\vec{r}')A_\beta(\vec{r}'). \tag{L236}$$

$$Q(\zeta,\vec{r}) \quad = \quad \frac{1}{\mathcal{V}}\sum_{\vec{k}} \delta(\zeta_{\vec{k}}-\zeta)e^{-i\vec{k}\cdot\vec{r}} \approx \frac{D(\mathcal{E}_F)\sin\sqrt{2m\zeta/\hbar^2}\,r}{2k_F r}, \tag{L237}$$

$$\sigma_{\alpha\beta}(\zeta,\zeta',\vec{R}) \quad = \quad -\frac{2\pi\hbar}{2}\left(\frac{e\hbar}{m}\right)^2 \frac{\partial}{\partial a_\alpha}\frac{\partial}{\partial a_\beta} Q(\zeta,\vec{R}-\vec{a})Q(\zeta',-(\vec{R}+\vec{a})\Big|_{\vec{a}=0} \tag{L238}$$

$$\approx \quad e^2\frac{v_F}{2\pi}D(\mathcal{E}_F)\frac{R_\alpha R_\beta}{R^4}\cos\left[\frac{(\zeta-\zeta')R}{\hbar v_F}\right]. \tag{L239}$$

Derivation of Meissner Effect

$$j^1_\alpha(\vec{r}) = \sum_\beta \int d\vec{r}' S^1_{\alpha\beta}(\vec{r}-\vec{r}')A_\beta(\vec{r}'), \tag{L240}$$

$$S^1_{\alpha\beta}(\vec{R}) = \frac{3ne^2}{4\pi^2 mc\hbar v_F}\frac{R_\alpha R_\beta}{R^4}\int d\zeta\, d\zeta' L(\zeta,\zeta')\cos\left[\frac{(\zeta-\zeta')R}{\hbar v_F}\right]. \tag{L241}$$

$$\xi = \frac{\hbar v_F}{\pi\Delta}. \tag{L242}$$

$$j_\alpha(\vec{r}) = \sum_\beta \int d\vec{r}'\, S_{\alpha\beta}(\vec{r}-\vec{r}')A_\beta(\vec{r}') \tag{L243a}$$

with

$$S_{\alpha\beta}(\vec{R}) = \frac{-3ne^2}{4\pi mc\xi}\frac{R_\alpha R_\beta}{R^4} I(R) \tag{L243b}$$

and

$$I(R) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dy'}{\cosh y'}\exp\left[-\frac{2R}{\pi\xi}\cosh y'\right]. \tag{L243c}$$

$$\vec{j}(\vec{r}) = \frac{-ne^2}{mc}\vec{A}(\vec{r}). \tag{L244}$$

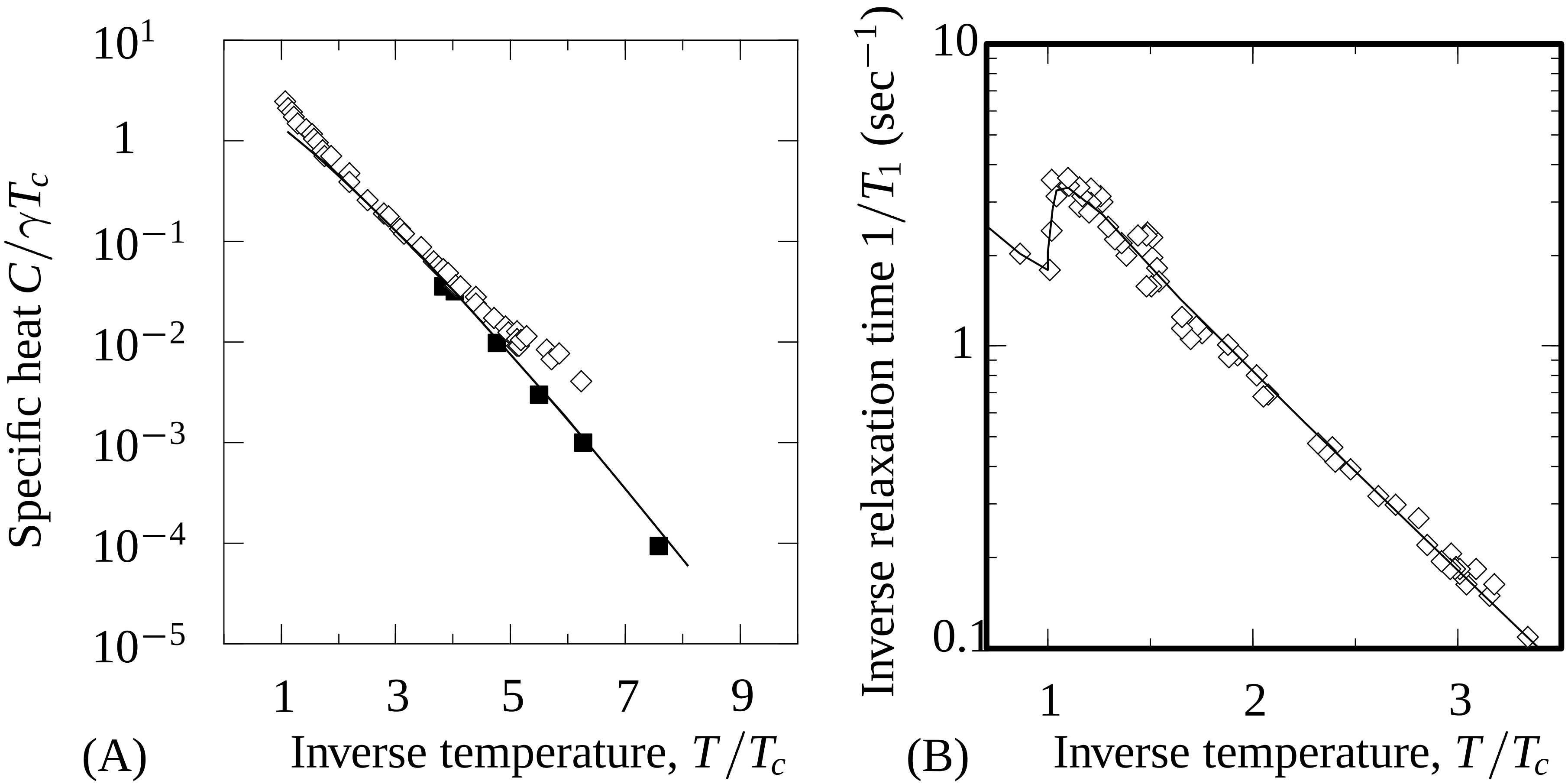


Figure 11: (A) Specific heat of aluminum and vanadium, relative to γT_c, where γ is the Sommerfeld parameter. [Boorse (1959)] (B) Inverse nuclear spin relaxation in aluminum compared with prediction of Bardeen, Cooper, and Schrieffer. [Masuda and Redfield (1962),]

Comparison with Experiment

$$\lambda_{\text{ep}} = -D(\mathcal{E}_F)\left\langle \frac{2|C_{\vec{k}-\vec{k}'}|^2\hbar\omega_{\vec{k}-\vec{k}'}}{(\epsilon_{\vec{k}}-\epsilon_{\vec{k}'})^2-\hbar^2\omega^2_{\vec{k}-\vec{k}'}} \right\rangle, \quad \mu^* = D(\mathcal{E}_F)\left\langle \frac{4\pi e^2}{|\vec{k}-\vec{k}'|^2+\kappa_{\text{c}}^2} \right\rangle. \qquad \text{(L245)}$$

$$T_c = \frac{\Theta_D}{1.45}\exp\left\{-\left[\frac{(1+\lambda_{\text{ep}})}{\lambda_{\text{ep}}-\mu^*(1+0.62\lambda_{\text{ep}})}\right]\right\}, \qquad \text{(L246)}$$

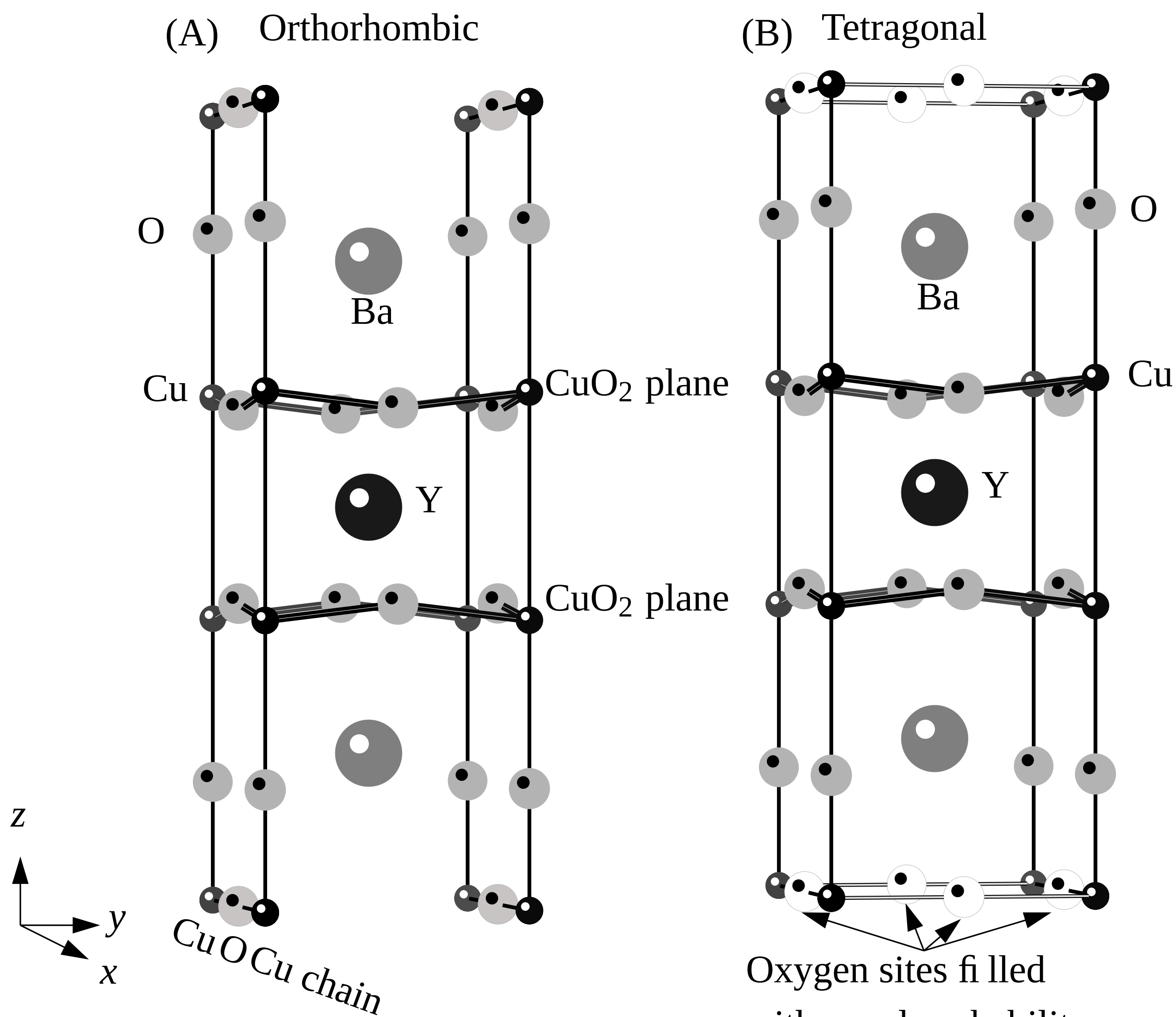


Figure 12: Structure of $YBa_2Cu_3O_x$, [Poole et al. (1988)] (A) Orthorhombic structure. (B) Tetragonal structure.

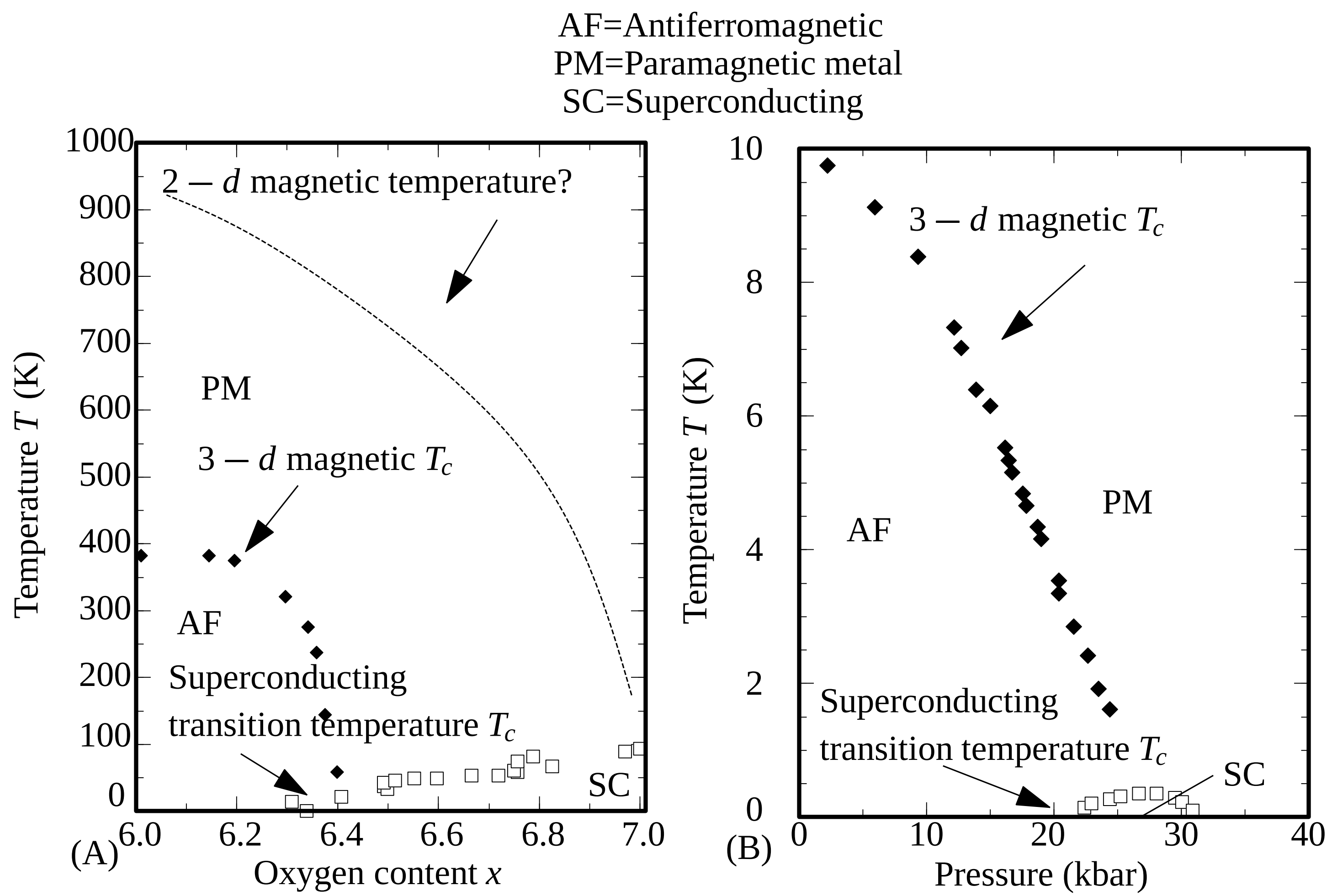


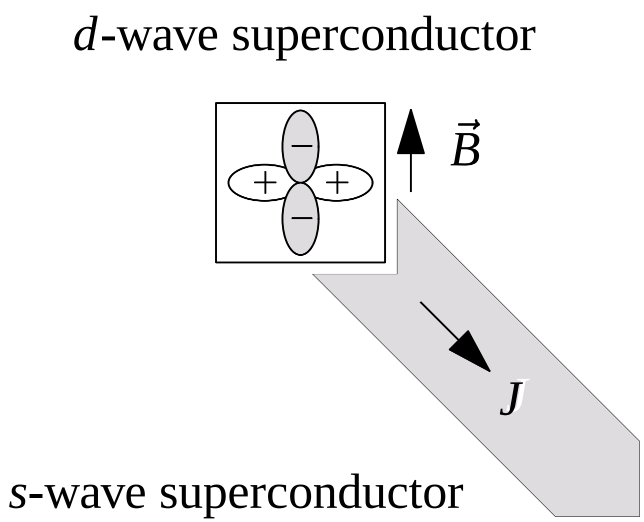
Figure 13: (A) Phase diagram for YBCO [Rossat-Mignod et al. (1990) and Greene and Bagley (1990)] (B) Heavy-fermion compound $CePd_2Si_2$.[Mathur et al. (1998).]

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\chi \text{ as } \Psi \rightarrow \Psi e^{-2ie\chi/\hbar c}. \tag{L247}$$

$$u_{\vec{k}}(\vec{r}) \rightarrow u_{\vec{k}}(\vec{r})e^{-ie\chi/\hbar c}, v_{\vec{k}}(\vec{r}) \rightarrow v_{\vec{k}}(\vec{r})e^{ie\chi/\hbar c}, \quad \text{and} \quad \Delta_{\vec{r}} \rightarrow \Delta_{\vec{r}}e^{-2ie\chi/\hbar c}. \tag{L248}$$

$$\Delta_{\vec{k}\vec{q}} = \sum_{\vec{k}'} \frac{U_{\vec{k}\vec{k}'}}{\mathcal{V}} \left\langle \hat{c}_{\vec{q}-\vec{k}'\downarrow}\hat{c}_{\vec{k}'\uparrow} \right\rangle. \tag{L249}$$

$$\Delta_{\vec{r}\vec{r}'} = \sum_{\vec{r}''} \frac{U_{\vec{r}\vec{r}''}}{\mathcal{V}} \left\langle \hat{c}_{-\vec{r}'\downarrow}\hat{c}_{\vec{r}''-\vec{r}'\uparrow} \right\rangle. \tag{L250}$$



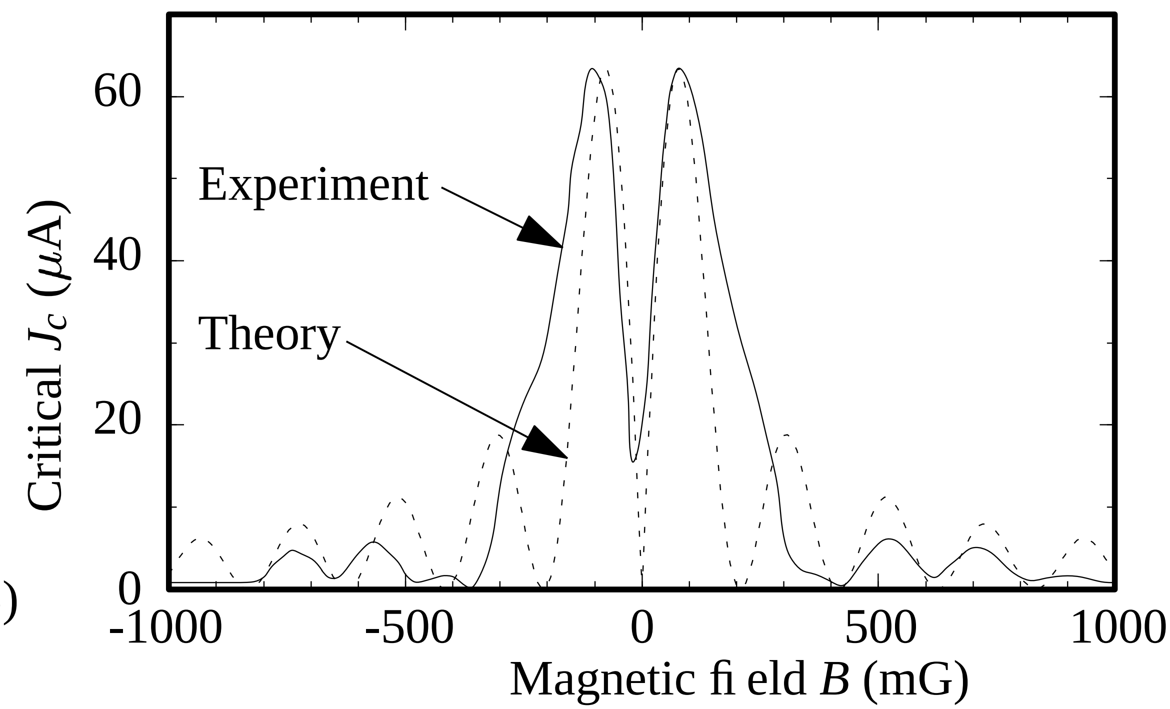


Figure 14: *d*–wave pairing. (A) Sketch of the experiment (B) Diffraction pattern. [Wollman et al. (1995)]