Physics 5413: Chaos, fractals, and nonlinear dynamics – Project 2

due date Friday, Feb 11, 2022

Biochemical switch (100 points)
(Problem 3.7.5 from Strogatz' book)

As one ingredient of biological pattern formation, Lewis and coworkers (1977) considered a model of a biochemical switch, in which a gene G is activated by a biochemical substance S to produce a pigment or other gene product. Let $g(t)$ denote the concentration of the gene product as a function of time. The model is given by

$$\frac{dg}{dt} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

where s_0 is the (time-independent) concentration of the substance S and the k's are positive constants. The first term describes the stimulation of the production of g by the substance S, the second is a linear degradation of g and the nonlinear term is an autocatalytic (positive) feedback.

1. Show that the system can be rewritten in dimensionless form as

$$dx/d\tau = s - rx + \frac{x^2}{(1 + x^2)}$$

   for suitably defined dimensionless quantities x, τ and parameters s, r.

2. Study the the model in the absence of stimulus (i.e. $s = 0$). Find all fixed points and analyze their stability. Show that some fixed points only exist if the degradation is not too strong, $r < r_c$. Find r_c.

3. Assume that initially there is no gene product, $g(0) = 0$, and suppose s is slowly increased from zero (the activating signal is turned on). What happens to $g(t)$? What happens if s then goes back to zero; does the gene turn off again?

4. Consider the model with stimulus ($s \neq 0$). Study the limiting cases of small and large s. Which terms in the differential equation do you need to keep? Find the fixed points in these limiting cases.

5. Now study the model numerically. This can be done either using a package like MathLab or by writing a program that integrates the differential equation. Explore the entire (r, s) parameter space and give a quantitatively accurate plot of the stability diagram.

   Hint: Follow the fixed points found in 2. with increasing s.