Physics 5413: Chaos and Dynamics – Project 4a

due date: Friday, March 11, 2022

Predator - Prey Model of Odell (100 points + 10 bonus points)
(Problem 8.2.8 from Strogatz’ book)

Consider the following model for the population dynamics of two species, a predator and a prey species (Odell 1980):

\[ \dot{x} = x(x(1-x) - y) \]
\[ \dot{y} = y(x - a) \]

where \( x \geq 0 \) is the dimensionless population of the prey and \( y \geq 0 \) is the dimensionless population of the predator. \( a \geq 0 \) is a control parameter.

1. What is the biological meaning of the parameter \( a \)?
2. Find the fixed points, study their stability and discuss their biological meaning.
3. Sketch the state space trajectories for \( a > 1 \), and show that the predators go extinct.
4. Show that a Hopf bifurcation exists at \( a_c = 1/2 \). Is it subcritical or supercritical?
5. Estimate the frequency of the limit cycle oscillations for \( a \) near the bifurcation.
6. Sketch state space trajectories for all qualitatively distinct cases for \( 0 < a < 1 \).

BONUS problem: Infinite-period bifurcation (10 BONUS points)

The Hopf bifurcation is not the only way to destroy a limit cycle. In this problem you will explore a different type of limit-cycle bifurcation not covered in class. Consider the following system:

\[ \dot{x}_1 = -\mu x_2 + x_1(1 - x_1^2 - x_2^2) - \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} \]
\[ \dot{x}_2 = \mu x_1 + x_2(1 - x_1^2 - x_2^2) - \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} \]

where \( \mu \) is the control parameter.

1. Identify fixed points and limit cycles. Use whatever method comes handy: numerical integration of the equations of motion, constructing a trapping region, or a direct analytical analysis. You may want to transform to polar coordinates \((r, \Theta)\) with \( x_1 = r \cos \Theta, \ x_2 = r \sin \Theta \).
2. Show that a bifurcation exists at \( \mu = 1 \). Describe the behavior above and below the bifurcation. Sketch the state space trajectories for both cases.
3. Why is this bifurcation called an infinite-period bifurcation?