Physics 5413: Chaos and Dynamics – Project 4b

due date Oct 12, 2018

Exploring chaos in the Lorenz model (100 points + 10 Bonus)

This is a mostly computational project. As before, you are encouraged to write your own computer programs. If you are not comfortable doing so you can use the two programs provided on the course web site. (They should do the job albeit in a very spartan form).

Consider the Lorenz model of convection rolls in the atmosphere:

$$\dot{x} = p(y - x) \tag{1}$$
$$\dot{y} = rx - y - xz \tag{2}$$
$$\dot{z} = xy - bz \tag{3}$$

where x, y, z are the dynamic variables and p, b and the Rayleigh number r are control parameters. p and b will be fixed at $p = 10$ and $b = 8/3$.

1. (optional) Write a computer program which integrates the Lorenz equations from $t = 0$ to $t = t_{\max}$ (with time step Δt) starting at (x_0, y_0, z_0) and records the trajectory. Due to the expected sensitivity of the dynamics to small perturbations, a good quality integrator should be used. A 4th order Runge-Kutta method is recommended. (If unclear, talk to instructor!)

2. Find out what time step Δt and time interval $t_{\max}$ to use for your computer experiments. Questions to think about: What are the intrinsic time scales of the problem? How long are the transients? You may want to do a few preliminary runs at typical parameter values, e.g. $r = 28$. What are reasonable starting values x_0, y_0, z_0?

3. Calculate a few trajectories for the following parameter values:
   $r = 10, 21, 24.5, 28, 100, 150, 166.3, 212, 400$.
   What do you observe? Plot both the time evolution of the variables and state space plots (either three-dimensional or suitable projections).

4. Follow the time evolution of two nearby trajectories. Plot the time evolution of the distance between the two points (what type of plot?) and calculate the Lyapunov exponents for $r = 10, 21, 28$. What are suitable initial separations? Over which time interval do you want to measure?

5. Horseshoe dynamics: For $r = 28$, investigate a set of 1000 trajectories with initial conditions inside a small cube of linear size Δx_0. Demonstrate how this cube is stretched, compressed and folded over time. For a very impressive picture, plot the positions of these 1000 trajectories at discrete times over a trajectory on the Lorenz attractor.

   What is a good value for Δx_0? How long does it take the "cube" to spread all over the Lorenz attractor? Now use a cube which is a thousand times smaller (if the cube represents a measurement error of the initial conditions in the atmosphere this would be a very impressive improvement). How long does it take now to completely spread the cube? Check a few more sizes for the cube. Can you guess the relationship between its size and the time the spreading takes? Make the appropriate plot.

6. (10 BONUS points) Investigate the dynamics for $145 < r < 166$. You will find a period doubling sequence. Compare with Feigenbaum's predictions.