Fireflies and frequency locking (100 points, adapted from Strogatz Sec 4.5)

Fireflies provide a spectacular example of frequency locking in nature. In this project, you will study a simple mathematical model of two fireflies trying to synchronize their flashes. The flashes of each firefly are controlled by an internal clock, i.e., by a phase variable $\phi$ that increases linearly with time according to $\dot{\phi} = \omega$ where the internal frequency $\omega$ is a constant that varies slightly from firefly to firefly. The firefly flashes whenever $\phi$ is an integer multiple of $2\pi$.

Two fireflies influence each other such that firefly 1 speeds up if it is behind and slows down if it is ahead (the same holds for firefly 2). This can be modeled by adding a periodic function $f$ of the phase difference to the equations for $\dot{\phi}$. The equations governing the flashes of two fireflies thus read

$$\begin{align*}
\dot{\phi}_1 &= \omega_1 + Af(\phi_2 - \phi_1), \\
\dot{\phi}_2 &= \omega_2 + Af(\phi_1 - \phi_2).
\end{align*}$$

Here, $A$ is the strength the coupling between the two fireflies, and $f$ is a triangle wave defined by

$$f(\Theta) = \begin{cases} 
\Theta & -\pi/2 \leq \Theta < \pi/2 \\
\pi - \Theta & \pi/2 \leq \Theta < 3\pi/2
\end{cases}$$

and extended periodically.

1. Derive an equation of motion for the phase difference between the two fireflies. Bring it into a dimensionless form such that it only depends on a single control parameter $\mu = (\omega_1 - \omega_2)/A$.

2. Find the range of entrainment, i.e., the range of $\mu$ for which the frequencies of the two fireflies lock.

3. If the fireflies frequency-lock, what is the phase difference between their flashes? Find the common period of their oscillations (the time between consecutive flashes of the same firefly).

4. If $\mu$ is outside the range of entrainment, the phase difference between the fireflies will drift. Find the time $T_{\text{drift}}$ it takes the phase difference to change by a full period of $2\pi$. 