Physics 5413: Chaos and Dynamics – Project 5

due date Mar 25, 2022

Fireflies and frequency locking (100 points, adapted from Strogatz Sec 4.5)

Fireflies provide a spectacular example of frequency locking in nature. In this project, you will study a simple mathematical model of two fireflies trying to synchronize their flashes.

The flashes of each firefly are controlled by an internal clock, i.e., by a phase variable ϕ that increases linearly with time according to $\dot{\phi} = \omega$ where the internal frequency ω is a constant that varies slightly from firefly to firefly. The firefly flashes whenever ϕ is an integer multiple of 2π.

Two fireflies influence each other such that firefly 1 speeds up if it is behind and slows down if it is ahead (the same holds for firefly 2). This can be modeled by adding a periodic function f of the phase difference to the equations for $\dot{\phi}$. The equations governing the flashes of two fireflies thus read

$$\begin{aligned} \dot{\phi}_1 &= \omega_1 + A\, f(\phi_2 - \phi_1) \ , \\ \dot{\phi}_2 &= \omega_2 + A\, f(\phi_1 - \phi_2) \ . \end{aligned}$$

Here, A is the strength the coupling between the two fireflies, and f is a triangle wave defined by

$$f(\Theta) = \begin{cases} \Theta & -\pi/2 \le \Theta < \pi/2 \\ \pi - \Theta & \pi/2 \le \Theta < 3\pi/2 \end{cases}$$

and extended periodically.

1. Derive an equation of motion for the phase difference between the two fireflies. Bring it into a dimensionless form such that it only depends on a single control parameter $\mu = (\omega_1 - \omega_2)/A$.
2. Find the *range of entrainment*, i.e., the range of μ for which the frequencies of the two fireflies lock.
3. If the fireflies frequency-lock, what is the phase difference between their flashes? Find the common period of their oscillations (the time between consecutive flashes of the same firefly).
4. If μ is outside the range of entrainment, the phase difference between the fireflies will drift. Find the time T_{drift} it takes the phase difference to change by a full period of 2π.