Physics 5413: Chaos and Dynamics – Project 7

due April 29, 2022

A strange repeller for the tent map (100 pts + 5 bonus, after Strogatz 11.4.6)

The tent map is defined on the closed interval [0,1] via $x_{n+1} = f(x_n)$, where

$$f(x) = \begin{cases} rx & (x \leq 1/2) \\ r(1-x) & (x \geq 1/2) \end{cases} .$$

Here, we assume that the control parameter $r > 2$. Then some points get mapped outside the interval [0,1]. If we start with $x_0 \in [0, 1]$ and $f(x_0) > 1$, then we say that x_0 has escaped after one iteration. Similarly, x_0 has escaped after n iterations, if $f^{(k)}(x_0) \in [0, 1]$ for all $k < n$ but $f^{(n)}(x_0) > 1$.

a) Find the set of initial conditions x_0 that escape after one iteration. Determine those that escape after two, three, and four iterations. Find the pattern.

b) Describe the set of x_0 that never escape, the so-called invariant set. Illustrate this set by making a qualitative plot. The invariant set is called a strange repeller because it has fractal structure and it repels all points not in the set.

c) Calculate the box dimension of the invariant set. Does it depend on r? How?

d) Show that the local Lyapunov exponent is positive at each point of the invariant set.

e) Is the invariant set a multifractal? Evaluate the generalized dimensions D_q. (You can assume the point density to be constant here. It will be evaluated in the bonus part.)

BONUS: Density of points

f) Consider the case $r = 2$. Show that the flat distribution $P(x) = 1$ (for $0 \leq x \leq 1$) of points x is invariant under the tent map. (This means that $P(y) = 1$ with $y = f(x)$).

g) How does this change for $r > 2$? Find the density of (surviving) points after two, three, four, ... iterations. Find the pattern and generalize to infinite iterations.