Physics 413: Statistical Mechanics - Final exam

Problem 1: Spin-3/2 paramagnet (80 points)
A paramagnetic material contains N localized (and thus distinguishable), non-interacting $S = \frac{3}{2}$ quantum spins in a magnetic field $\vec{B} = B\mathbf{e}_z$ in z-direction. The Hamiltonian is given by

$$H = -\mu B \sum_{i=1}^{N} S_i^z,$$

where S_i^z has the eigenvalues $-3/2, -1/2,\ 1/2,\ 3/2$. Here, μ is the magnetic moment associated with the spins.

a) Qualitatively discuss the ground state. What is the ground state energy E_0? What is the ground state magnetization $M_0 = \mu \sum_{i=1}^{N} S_i^z$? (10 points)

b) Use the canonical ensemble to calculate the partition function and the Helmholtz free energy. (20 points)

c) Compute the internal energy as a function of temperature and discuss its behavior in the limits $k_B T \ll \mu B$ and $k_B T \gg \mu B$. Compare with the ground state energy you found in a). (20 points)

d) Calculate the magnetization $M = \mu \sum_{i=1}^{N} \langle S_i^z \rangle$ as function of temperature and field. Discuss its behavior in the limits $k_B T \ll \mu B$ and $k_B T \gg \mu B$. Compare with the ground state magnetization you found in a). (20 points)

e) Find the magnetic susceptibility $\chi = (\partial M / \partial B)_T$ in the high temperature limit $k_B T \gg \mu B$ and determine the Curie constant. [Hint: You can simplify the calculation by taking the high-temperature limit before calculation the field-derivative.] (10 points)

Problem 2: Debye phonons in one dimension (80 points)
Consider a one-dimensional solid of N_A atoms and length L. This solid has longitudinal phonons but no transversal ones. Within the Debye model, the phonons have frequencies $\omega_k = c|k|$ (where c is the sound velocity and k is the wave number). Phonon modes only exist for frequencies below the Debye frequency Ω_D.

a) Calculate the density of states $g(\epsilon)$. (20 points)

b) What is the total number of phonon modes in a chain of N_A atoms? Use this to determine the Debye frequency Ω_D as a function of N_A and L. (20 points)

c) Derive an expression for the internal energy as a functions of temperature. (20 points)

d) Discuss the internal energy in the limits of high ($k_B T \gg \hbar\Omega_D$) and low ($k_B T \ll \hbar\Omega_D$) temperatures. Compare with the equipartition theorem. (20 points)

Problem 3: Mean-field theory of an Ising model with long-range interactions (90 points + 25 BONUS)

Consider a one-dimensional Ising model of N spins, $S_i = \pm 1$, given by a Hamiltonian

$$H = -J \sum_{i \neq j} \frac{1}{(i-j)^2} S_i S_j - \mu B \sum_i S_i$$

with a positive interaction constant J (i.e., the interaction is *not* just between nearest neighbors but between all pairs; its magnitude decays quadratically with distance). Assume $N \gg 1$ and periodic boundary conditions.

a) Qualitatively discuss the ground state in the absence of a field ($B = 0$)? What is its energy? (10 points)

b) Derive the mean-field Hamiltonian for this model. (20 points)

c) Solve the mean-field Hamiltonian and derive the mean-field equation for the magnetization. (20 points)

d) Find the critical temperature for the onset of magnetic order in the absence of a field. (20 points)

e) Determine the magnetic susceptibility above the critical temperature in the limit of small fields. Is the Weiss temperature positive or negative. (20 points)

f) [25 BONUS points] Consider the case $J < 0$. Qualitatively describe the ground state and calculate its energy. Derive the mean-field theory for the type of order you have identified. Find the critical temperature.

$\int_0^\infty dx\; x/(e^x - 1) = \pi^2/6$.

$\tanh(x) = x - \frac{1}{3}x^3 + ...,\quad \mathrm{Artanh(x)} = \mathrm{x} + \frac{1}{3}\mathrm{x}^3 + ...$

$\sum_{n=1}^{\infty}(1/n^2) = \pi^2/6,\quad \sum_{n=1}^{\infty}[(-1)^n/n^2] = -\pi^2/12$