Physics 413: Statistical Mechanics - Final exam

Problem 1: Classical ideal gas in a cylindrical trap (80 points)

Consider a three-dimensional non-relativistic classical ideal gas of N (indistinguishable) particles of mass m at temperature T. It is held in a cylindrical trap with hard walls at $z = 0$ and $z = H$ and a harmonic confining potential in the x and y directions. In other words, the trap potential reads

$$V(\vec{r}) = \begin{cases} \frac{a}{2}(x^2 + y^2) & \text{for } \ 0 < z < H \\ \infty & \text{otherwise} \end{cases}$$

where $a > 0$ is a constant.

a) Using the canonical ensemble, calculate the partition function and the Helmholtz free energy of the gas. (20 points)

b) Determine the internal energy and the specific heat. Compare your results with the equipartition theorem. (20 points)

c) Find the spatial particle density (particles per volume) $n(\vec{r})$ in the trap. (20 points)

d) Find the size of the particle cloud in the x and y directions defined as the mean-square distance $\langle x^2 \rangle = \langle y^2 \rangle$ of the particles from the cylinder axis. (20 points)

Problem 2: Mean-field theory of a spin-1 ferromagnet (100 points)

A model of a spin-1 ferromagnet on a cubic lattice is given by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i S_j - \mu B \sum_i S_i$$

with a positive interaction constant J between nearest neighbors. The N spin variables S_i each can take **three** different values $S_i = -1, 0, 1$.

a) Qualitatively discuss the ground state in the limits $\mu B \gg J$ and $\mu B \ll J$. (10 points)

b) Derive the mean-field Hamiltonian for this model. (20 points)

c) Solve the mean-field Hamiltonian and derive the mean-field equation for the magnetization m. (20 points)

d) Find the critical temperature for the onset of ferromagnetism (in the absence of a field). (15 points)

e) Expand the mean-field equation about the critical point to linear order in B and to cubic order in m. (20 points)

f) Determine the critical exponents β (magnetization), γ (susceptibility), and δ (critical isotherm). (15 points)

Problem 3: Bose-Einstein condensation (70 points)

Consider N noninteracting bosons with spin 1 and dispersion relation $\epsilon(\vec{k}) = a|\vec{k}|^{3/2}$ on a planar square surface of linear size L.

a) Find the density of states $g(\epsilon)$. (20 points)

b) Does the system undergo Bose-Einstein condensation? Why? (10 points)

c) Find the Bose-Einstein condensation temperature, if any, as a function of the particle density N/L^2. (20 points)

d) Calculate the internal energy and the specific heat as functions of temperature in the condensed phase. (20 points)

Bonus Problem 4: Quantum Griffiths phase (80 points replacing your lowest mid-semester score)

A magnet in the quantum Griffiths phase contains a large number N of magnetic clusters of random volume V_i. The probability density of the volume is given by $P_V(V) = \mathcal{N}\exp(-cV)$ where c is a system parameter and $\mathcal{N}$ is the normalization constant. Each cluster acts as a two-level system with energies 0 and ϵ_i. The energy ϵ_i depends on the volume of the cluster via $\epsilon_i = \epsilon_0 \exp(-aV_i)$ where ϵ_0 and a are constants.

a) Consider a single two-level system. Using the canonical ensemble, calculate the internal energy and the specific heat of the cluster as a function of ϵ_i and the temperature T. (25 points)

b) Find the normalization constant $\mathcal{N}$. (5 points)

c) Calculate the density of states of the clusters, i.e., the probability density $P_\epsilon(\epsilon)$ of the cluster energies. (25 points)

d) Now consider the entire magnet. Determine the total contribution of all clusters to the internal energy $U_{\text{tot}}(T)$ and to the specific heat $C_{\text{tot}}(T)$ as functions of temperature for low but nonzero temperatures ($\epsilon_0 \gg k_B T$). The sum over the clusters can be replaced by an integral over the density of states, $\sum_{i=1}^{N} \to N \int d\epsilon\, P_\epsilon(\epsilon)$. (25 points)

$dU = TdS + \delta W + \mu dN \qquad \ln(N!) \approx N\ln(N) - N$

$\sinh(x) = x + \frac{1}{6}x^3 + \ldots, \quad \cosh(x) = 1 + \frac{1}{2}x^2 + \ldots, \quad \tanh(x) = x - \frac{1}{3}x^3 + \ldots, \quad \frac{1}{1+x} = 1 - x + x^2 \pm \ldots$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2/\sigma^2} = (2\pi\sigma^2)^{1/2} \qquad \int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{1}{2}x^2/\sigma^2} = (2\pi\sigma^2)^{1/2}\sigma^2$$

$$\int_0^{\infty} dx \frac{x^\alpha}{e^x - 1} = \Gamma(1+\alpha)\zeta(1+\alpha) \qquad (\text{for } a > 0)$$

$$\int_0^{\infty} dx \frac{x^\alpha}{e^x + 1} = (1 - 2^{-\alpha})\Gamma(1+\alpha)\zeta(1+\alpha) \qquad (\text{for } a > -1)$$