Physics 6311: Statistical Mechanics - Final Exam

May 15, 2019

Problem 1: Two-level systems (50 pts)

a) Consider a single two-level system with states at energies 0 and $\epsilon$. Use the canonical ensemble to calculate its specific heat as function of temperature. (30 pts)

b) A piece of material contains a large number $N$ of such two-level systems. Their energies $\epsilon$ are randomly distributed between 0 and $\epsilon_{\text{max}}$ with a probability density $P(\epsilon) = C\epsilon^{\lambda-1}$ characterized by a positive exponent $\lambda$ ($C$ is the normalization constant). Determine the average specific heat of the entire sample for temperatures $k_B T \ll \epsilon_{\text{max}}$. (You do not need to evaluate constants given by dimensionless integrals). (20 pts)

Problem 2: Fermi gas with quartic energy-momentum relation (50 pts)

Consider a gas of $N$ noninteracting spin-1/2 fermions at zero temperature in a cubic box of linear size $L$. The single-particle energies of these fermions are given by $\epsilon(k) = A|k|^4$ where $k$ is the wave vector and $A$ is a constant.

a) Find the Fermi energy $\epsilon_F$ as function of the density $N/L^3$. (25 pts)

b) Determine the total internal energy as a function of $N$ and $L$. (25 pts)

Problem 3: Bose-Einstein condensation with absorption sites. (80 pts)

An ideal Bose gas of $N$ nonrelativistic spin-0 particles of mass $m$ is in a cubic box of linear size $L$. In addition, there are $N_A \ll N$ absorption sites on the surfaces of the box. Each absorption site can either be empty, or contain a single of the Bose particles. An absorbed particle has energy $\Delta$.

The absorbed particles are in equilibrium with the particles in the gas.

a) Starting from the Bose distribution, find the Bose-Einstein condensation temperature when no absorption sites are present, i.e., for $N_A = 0$. (You do not need to evaluate constants given by dimensionless integrals). (35 pts)

b) What is the value of the chemical potential $\mu$ of the Bose gas in the condensed phase? (5 pts)

c) Using the grand canonical ensemble, calculate the average number of absorbed particles as a function of temperature $T$ and energy $\Delta$, at this chemical potential $\mu$. (20 pts)

d) Describe how does the critical temperature for Bose-Einstein condensation change due to the absorption sites? Find its dependence on $N_A$ and $\Delta$. (Hint: The total particle number is the sum of the number of particles in the gas and the number of absorbed particles.) (10 pts)

e) Discuss the limits $\Delta \to -\infty$ and $\Delta \to \infty$. (10 pts)
Problem 4: Mean-field theory of Blume-Capel model (120 pts)

Each site of a square lattice is occupied by a spin 1, i.e., by a variable \( S_i \) that can take values \(-1, 0, +1\). The Hamiltonian reads

\[
H = -J \sum_{\langle ij \rangle} S_i S_j + \Delta \sum_i S_i^2 - \mu_B B \sum_i S_i
\]

where the first sum runs over all pairs of nearest neighbors and \( J > 0 \). The so-called crystal field energy \( \Delta \) can take positive or negative values. This Hamiltonian is called the Blume-Capel model.

a) Analyze the system at zero temperature and zero magnetic field: What is the ground state for negative \( \Delta \)? What is the ground state for large positive \( \Delta \gg J \)? Compute the ground state energies. (20 pts)
b) The system undergoes a phase transition as a function of \( \Delta \) at fixed \( J \), zero temperature, and zero magnetic field. At what value of \( \Delta \) does the transition happen? Is the transition continuous or of first order? (10 pts)
c) Derive a mean-field approximation of the Hamiltonian. (20 pts)
d) Solve the mean-field Hamiltonian and derive the mean-field equation. (25 pts)
e) Solve this mean-field equation for \( B = 0 \) and find the critical temperature \( T_c \) as function of \( J \) and \( \Delta \). (You do not need to solve the final transcendental equation for \( T_c \).) (20 pts)
f) Discuss how \( T_c \) behaves in the limits \( \Delta \to \infty \) and \( \Delta \to -\infty \). (15 pts)
g) Describe how you would decide whether the transition is continuous or of first order. You do not actually have to perform the calculation. (10 pts)