# Physics 6311: Statistical Mechanics - Test Preparation Homework 13

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due date: Thursday Dec 9, 2021

**Problem 1: Random boxes** (10 points)

A machine in a factory making cylindrical boxes is malfunctioning. As a result, it is producing cylinders of random size. Specifically, the diameter and the height of the cylinder are independent random quantities. They can take values between 0 and 2 m with a constant probability density.

a) What are the maximum and minimum possible values for $V$?
b) Calculate the average volume $\langle V \rangle$ of the produced cylinders and its standard deviation.
c) Derive the probability density of $V$. (Hint: Be careful with the integration bounds when transforming and integrating over the $\delta$-function)
d) What is the most likely volume?

**Problem 2: Two interacting spin-1/2 particles** (10 points)

Consider two interacting Ising spins $S_1$ and $S_2$ which each can take the values $+1$ or $-1$. The Hamiltonian reads

$$H = -JS_1S_2$$

where the interaction energy $J$ is a positive constant.

a) Write down all microstates of the system.

b) Using the canonical ensemble, calculate the partition function and the Helmholtz free energy of the gas.

c) Determine the internal energy and the specific heat.

d) Find the entropy.

e) Discuss the high and low-temperature limits of the entropy.

**Problem 3: Debye phonons in two dimension** (10 points)

Consider a thin film (two-dimensional solid) of $N$ atoms and linear size $L$. This solid has 3N phonon modes. Within the Debye model the phonons with frequencies $\omega_{\vec{k}} = c|k|$ (where $c$ is the speed of sound) exist for $\omega_{\vec{k}} < \Omega_D$.

a) Calculate the density of states $g(\epsilon)$.

b) Determine the Debye frequency $\Omega_D$.

c) Calculate the internal energy and the specific heat for low temperatures ($k_B T \ll \hbar\Omega_D$).

**Problem 4: Fermi gas**

Consider a gas of $N$ non-interacting spin-1/2 fermions on a square surface of linear size $L$. They feature the usual nonrelativistic dispersion relation $\epsilon(\vec{k}) = \hbar^2 k^2/(2m)$.

a) Find the Fermi momentum and Fermi energy as functions of $N$ and $L$.

b) Determine the total ground state energy.

c) Find the zero-temperature pressure.