Physics 6311: Statistical Mechanics - Homework 3

due date: September 14, 2021

Problem 1: Probability of a density fluctuation (10 points)

Consider two identical boxes, A and B.

a) 10 particles are distributed over the two boxes at random. Calculate the probabilities $P(4)$ and $P(5)$ for finding exactly $N_A = 4$ and $N_A = 5$ particles in the box A, respectively. Calculate $P(4)/P(5)$.

b) Repeat the calculations for $10^{23}$ particles. Compare $N_A = 0.4 \times 10^{23}$ and $N_A = 0.5 \times 10^{23}$.

(For large $n$ the factorial can be approximated by Stirling’s formula $n! \approx \sqrt{2\pi n} \ n^n e^{-n}$)

Problem 2: Transformation of random variables (10 points)

The random variables $X$ and $Y$ are independent and Gaussian distributed with averages $\langle x \rangle = \langle y \rangle = 0$ and standard deviations $\sigma_x = \sigma_y = 1$.

a) Calculate the characteristic functions of $X$ and $Y$.

b) Find the characteristic function for the random variable $Z = X^2 + Y^2$.

c) Derive the probability density $P_Z(z)$ of the random variable $Z$.

d) Compute the moments $\langle z \rangle, \langle z^2 \rangle$, and $\langle z^3 \rangle$. Find the first three cumulants.

Problem 3: Power-law distribution (10 point)

The probability density of a random variable $J$ has the power-law form

$$P_J(j) = \begin{cases} A (1 - j)^y & (0 \leq j \leq 1) \\ 0 & \text{otherwise} \end{cases},$$

where $A$ and $y$ are constants.

a) Find the normalization constant $A$ as a function of the exponent $y$. For which $y$ is $P_J$ a valid probability density?

b) Calculate the average of $J$ as well as the geometric mean. How do they compare for $y \to -1$?

Problem 4: Diode (10 points)

The current $I$ across a diode is related to the applied voltage $V$ via

$$I = I_0 [e^{eV/(k_BT)} - 1].$$

The diode is subject to a random voltage $V$ which is Gaussian distributed with zero mean and variance $\sigma^2$. Calculate the probability density $P(I)$ of the current. Find the most probable current and the average current.