Physics 6311: Statistical Mechanics - Homework 5

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due date: Tuesday, Sep 28, 2021

Problem 1: Spin-$\frac{1}{2}$ in a magnetic field (15 points)

Consider N (distinguishable) $S = \frac{1}{2}$ quantum spins in a magnetic field $\mathbf{B} = B\mathbf{e}_z$ in z-direction. The Hamiltonian is given by

$$\hat{H} = -Bg\mu_B \sum_{i=1}^{N} S_i^{(z)}, \qquad S_i^{(z)} \pm \frac{1}{2}.$$

Here g is the gyromagnetic factor and μ_B is the Bohr magneton. The spins are coupled to a heat bath at temperature T.

a) Use the canonical ensemble to calculate the partition function, Helmholtz free energy, the entropy, the internal energy and the specific heat as functions of temperature.

b) Calculate the magnetization $M = g\mu_B \sum_{i=1}^{N} \langle S_i^{(z)} \rangle$ and the magnetic susceptibility $\chi = (\partial M/\partial B)_T$ as functions of T and B. Determine the zero-field susceptibility $\lim_{B\to 0} \chi(B,T)$.

c) For fixed $B > 0$, at what temperature is the maximum of the specific heat (the so-called Schottky peak)?

Problem 2: Quantum harmonic oscillators in canonical ensemble (10 points)

Consider a system of N independent one-dimensional quantum harmonic oscillators with energy levels $\epsilon_n = \hbar\omega n$ (neglecting the zero-point energy) in the canonical ensemble at temperature T.

a) Calculate the partition function and the Helmholtz free energy.

b) Compute the average energy and entropy as functions of N and T. Does the system fulfill the third law of thermodynamics?

c) Compute the heat capacity as function of temperature.

Problem 3: Two interacting magnetic moments in the canonical ensemble (15 points)

Consider two classical magnetic moments characterized by three-dimensional unit vectors $\mathbf{S}_1$ and $\mathbf{S}_2$. They interact via an exchange interaction with the Hamiltonian $H = -J\ \mathbf{S}_1 \cdot \mathbf{S}_2$ where J is a positive constant.

a) In the ground state, what do you expect the relative orientation of the two moments to be?

b) Use the canonical ensemble to calculate the partition function and the Helmholtz free energy.

c) Determine average energy and heat capacity as functions of temperature.

d) At low temperatures, what is the average angle between the moments, and what is its standard deviation?

Problem 3: Specific heat of an anharmonic oscillator (15 points)

An anharmonic *classical* oscillator has the Hamiltonian

$$H = \frac{p^2}{2m} + V_0 \cosh(x/x_0)$$

where the mass m as well as the potential parameters V_0 and x_0 are constants.

a) Calculate the specific heat as a power series in the temperature T to linear order in T. (To decide how to set up the expansion, think about where the particle will be at low and high temperatures, respectively. What does this mean for the potential?)

b) Also calculate the contribution of order T^2.