Physics 6311: Mid-semester test — Mar 12, 2019

Problem 1: Particle at random position (60 points)

Consider a particle in one dimension restricted to the positive x-axis. Its position x is a random variable with probability density $P_X(x) \sim \exp(-ax)$ where a is a positive constant.

a) Find the normalization constant of the probability density $P_X(x)$.

b) The potential energy of the particle is given by $V(x) = (1/2)kx^2$ with k being a constant. Calculate the average force acting on the particle.

c) Find the probability density $P_V(V)$ of the potential energy.

Problem 2: Ising spins in magnetic field (80 points + 10 bonus)

Consider two non-interacting, distinguishable Ising spins in a magnetic field described by the Hamiltonian $H = -h(S_1 + S_2)$, Here S_1, S_2 can take values ± 1 and h is a constant proportional to the magnetic field.

a) Using the canonical ensemble, find the partition function and the free energy as functions of temperature the T and h.

b) Calculate the average total magnetization $M = \langle S_1 + S_2 \rangle$ as well as the magnetic susceptibility $\chi = (\partial M/\partial h)_T$ at $h = 0$.

Now assume that the two spins have a ferromagnetic interaction $J > 0$, i.e., the Hamiltonian now reads $H = -h(S_1 + S_2) - JS_1S_2$.

c) Using the canonical ensemble, find the partition function and the free energy as functions of T and h (Hint: What are the possible states of the 2-spin system?)

d) Calculate M and χ for the interacting spin pair as in part b).

e) (10 bonus points) Discuss the behavior of χ for $J \ll k_BT$ and $J \gg k_BT$ and compare with the result in part b).

Problem 3: Ideal gas in a linear potential well (60 points)

Consider a non-relativistic classical ideal gas of N indistinguishable particles of mass m at temperature T in a three-dimensional potential well of the form $E_{pot}(\vec{r}) = B|\vec{r}|$ (B is a constant).

a) Calculate the canonical partition function and the free energy of the gas.

b) Determine the average energy and the specific heat. Compare with the equipartition theorem.

c) Calculate how the particle density $n(\vec{r})$ changes with $\vec{r}$. (Hint: the particle density $n(\vec{r})$ is a reduced probability density of the phase space density $\rho(\vec{r}, \vec{p})$.)

$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-x_0)^2/\sigma^2} = (2\pi\sigma^2)^{1/2}$

$\int_0^{\infty} dx\, x e^{-ax} = 1/a^2, \qquad \int_0^{\infty} dx\, x^2 e^{-ax} = 2/a^3, \qquad \int_0^{\infty} dx\, x^3 e^{-ax} = 6/a^4, \qquad \int_0^{\infty} dx\, x^4 e^{-ax} = 24/a^5$