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COMMENT

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In the context of the classical multivariate linear regression model, Cook, Li and Chiaromonte proposed a parsimonious parameterization using the novel concept of an "*envelope*", which yields an asymptotically less variable MLE comparing to the traditional multivariate linear regression estimator. One question that arises naturally is how this method can be generalized to deal with nonlinear or (and) nonnormal multivariate regression models. In this discussion, we explore the applicability of the *envelope* method when the mean function is nonlinear. Specifically, we assume the multivariate nonlinear regression model

$$Y^j = f_j(\alpha_j + \boldsymbol{\beta}_j \mathbf{X}) + \epsilon_j, \tag{1}$$

where $\mathbf{Y} = (Y^1, \ldots, Y^r)$ is the random response vector, $\mathbf{X}$ is a p-dimensional random vector of predictors, $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ is independent with $\mathbf{X}$ and is normally distributed with mean 0 and unknown covariance matrix $\boldsymbol{\Sigma}$, $f_j(.)$, $j = 1, \ldots, r$, are arbitrary unknown link functions. Both $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $\boldsymbol{\beta} \in \mathbb{R}^{r\times p}$ are unknown, while $\boldsymbol{\beta}_j$, $j = 1, \ldots, r$, is the jth row of $\boldsymbol{\beta}$. This is the same model that Cook and Setodji (2003) considered in the context of sufficient dimension reduction. Our goal is to estimate span($\boldsymbol{\beta}$), referred to as the *multivariate central mean subspace* in sufficient dimension reduction literature (Cook and Setodji 2003).

Assuming that $\mathbf{X}$ satisfies the so-called linearity condition (Li and Duan 1989), $\mathrm{E}(\mathbf{X}|\boldsymbol{\beta}\mathbf{X})$ is a linear function of $\boldsymbol{\beta}\mathbf{X}$, commonly used in sufficient dimension reduction methods and one which holds for elliptically contoured predictors

(Eaton (1986)), hence holds when $\mathbf{X}$ is multivariate normal, the following Lemma suggests that the *envelope* method that Cook et. al. proposed still holds under link violation. The justification of our Lemma is similar to that of Proposition 8.1 of Cook (1998) which stems from Theorem 2.1 of Li and Duan (1989).

Lemma 1. *If* (1) *holds and* $\mathrm{E}(\mathbf{X}|\boldsymbol{\beta}\mathbf{X})$ *is a linear function of* $\boldsymbol{\beta}\mathbf{X}$*, then* $\hat{\boldsymbol{\beta}}$*, the maximum likelihood estimator obtained via envelope method under the misspecified linear link function, is a Fisher consistent estimator of* $\boldsymbol{\beta}$ *up to a multiplicative scalar.*

We may also compare the envelope estimator with those obtained via sufficient dimension reduction methods assuming (1). We conjecture that it would perform better than those sufficient dimension reduction methods which are currently available for multivariate responses due to the property of MLE. As to the restriction of the Gaussian distribution for the multivariate linear model considered in Cook et al., we may be able to extend the envelope method to the natural exponential family. Further research along this line is underway.

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COMMENT

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We congratulate the authors for their path breaking work that provides us new insight into multivariate linear regression. By introducing the envelope models, the authors open up an avenue toward efficient redundancy reduction. Our