Entrainment and stimulated emission of ultrasonic piezoelectric auto-oscillators

Richard L. Weaver[a] and Oleg I. Lobkis
[1]Department of Physics, University of Illinois, 1110 W Green Street, Urbana, Illinois 61801, USA

Alexey Yamilov
[2]Department of Physics, University of Missouri-Rolla, 1870 Miner Circle, Rolla, Missouri 65409, USA

(Received 24 February 2007; revised 27 September 2007; accepted 28 September 2007)

Theoretical modeling and laboratory tests are conducted for nonlinear auto-oscillating piezoelectric ultrasonic devices coupled to reverberant elastic bodies. The devices are shown to exhibit behavior familiar from the theory of coupled auto-oscillators. In particular, these spontaneously emitting devices adjust their limit-cycle frequency to the spectrum of the body. It is further shown that the auto-oscillations can be entrained by an applied field; an incident wave at a frequency close to the frequency of the natural limit cycle entrains the oscillator. Special attention is paid to the phase of entrainment. Depending on details, the phase is such that the oscillator can be in a state of stimulated emission: the incident field amplifies the ultrasonic power emitted by the oscillator. These behaviors are essential to eventual design of an ultrasonic system that would consist of a number of such devices all synchronized to their mutual field, a system that would be an analog to a laser. A prototype *uaser* is constructed. © *2007 Acoustical Society of America.*. [DOI: 10.1121/1.2800315]

PACS number(s): 43.35.Yb, 43.58.Wc, 43.38.Ew [TDM] Pages: 3409–3418

I. INTRODUCTION

Auto-oscillators are common in acoustics. Simple autonomous nonlinear mechanical systems, with steady forcing, can have stable limit cycles. Examples are the bowing of a violin string[1] vortex shedding by a steady flow over an obstacle[2] or through a clarinet reed. Otoacoustic emissions[3] are of this class, as are thermo-acoustic engines.[4] Unstable feedback loops[5–7] occur in room acoustics, hearing aids[8] and vibration control.[9] The intriguing case in which two or more mechanical limit cycle oscillators are coupled through a linear acoustic medium has also received attention.[10–12] Here we study the case of an ultrasonic transducer as an element in a nonlinear auto-oscillating electronic circuit, its behavior with and without an acoustic medium, and its behavior when an incident acoustic field is applied. As might be expected, it is found that an incident acoustic field can entrain such an oscillator to its frequency. More significantly, it is further found that entrainment can occur with a phase that corresponds to an increased rate of ultrasonic energy emission from the transducer, i.e., to stimulated emission. In the absence of any applied field, a set of two or three such transducers is demonstrated to synchronize itself to the complex mutual radiation field, and to do so with phases that correspond to stimulated emission. Such a system may be thought of as an analog to a laser.

Lasing is not necessarily a quantum phenomenon. Classical laser designs[13–19] have been proposed but little has yet been realized in the laboratory. Here we report construction of nonlinear electronic oscillators coupled piezoelectrically to an elastic body and capable of both spontaneous and stimulated emission. Such systems may have both pedagogic and research value. Because acoustic waves with their longer wavelengths and longer time scales permit probes and controls to a degree not possible in optics, these systems may permit experiments that complement those possible with lasers.

The classical laser designs of Borenstein and Lamb[13,14] and Kobelev *et al.*[16] are composed of incoherently excited Duffing oscillators. The oscillators emit spontaneously in a trivial fashion. Theoretical arguments indicate that, when they go into resonance with each other with the right phase relation as enforced by nonlinear interactions with their mutual radiation field, they also emit by stimulated emission. An acoustic version has been reported by Bredikhin *et al.*[17] who present theory and measurements on a system of impulsively excited nonlinear bubbles. While ringing down, the bubbles sometimes synchronize by means of their nonlinearity and their mutual radiation field. The system differs from the present design based on limit cycle oscillators.

Zavtrak and co-workers[18,19] have suggested that bubbles or other particles in a fluid could be pumped by an applied coherent harmonic electric or acoustic field. (This differs from the usual notion of a laser as converting incoherent pump excitation to coherent radiation.) They suggest that the particles would bunch spatially under the influence of their radiation forces, leading to a coherent re-emission of sound in a direction imposed by their radiation field and the modes of their cavity. This acoustic version of a free-electron laser or gyrotron[20] has not been constructed.

Independently, recent years have seen considerable interest in dynamic synchronization[10,21–28] in which sets of distinct coupled auto-oscillating limit cycles synchronize to each other. The phenomenon occurs in disparate circumstances, including firefly flashes, brain waves, esophagal

[a]Author to whom correspondence should be addressed. Electronic mail: r-weaver@uiuc.edu

waves, bridges with crowds of pedestrians, and chemical oscillations. It occurs between lasers, thermo-acoustic engines, Josephson junctions, metronomes and pendulum clocks. The chief mathematical model for studying the synchronization of large numbers of auto-oscillators is that of Kuramoto, and its generalizations.[23–27] After arguing that the state of a limit cycle oscillator is well represented in terms of its phase ξ, and that the phase of each of the oscillators is weakly coupled to the phases of the others, Kuramoto[29] derived a set of N coupled Adler equations.

$$d\xi_n/dt = \omega_n + (1/N)\sum_{m=1}^{N} A \sin(\xi_m - \xi_n).$$

Kuramoto showed that this model, in the thermodynamic limit $N \to \infty$, exhibits a phase transition in which a macroscopic number of oscillators synchronize to each other if A is sufficiently large compared to the deviations among the many ω_n. Sundry generalizations have been discussed also, such as time delays and randomness in the couplings. Resemblance between this phase transition and the onset of coherence among the atoms of a laser has been noted.[22] In the event that coupling is mediated by a complex wave-bearing body, as in the experiments to be reported here and in large arrays of THz Josephson junctions[22,30] or the random laser,[31] the coupling is neither uniform as in the Kuramoto model above, nor completely random as in the model of Daido.[25] Thus such systems constitute a so-far unexplored model.

Synchronization of a set of oscillators is not sufficient for emulation of a laser. The sine-qua-non of a laser is *stimulated emission*, in which a wave incident upon an excited oscillator is re-emitted with unchanged phase and increased amplitude. This is a classical phenomenon,[13–19] and is describable without quantum mechanics. A classical linear oscillator will exhibit stimulated absorption or stimulated emission depending on the phase of the oscillation relative to that of the incident field. Thus an incoherent array of excited linear oscillators will show no net stimulated emission. If, however, all or most of the oscillators can be induced to have the same frequency and the correct phases, the set will exhibit stimulated emission. Only to the extent that oscillators and incident fields have the same frequency and the correct phase difference will the energy in the oscillators be transferred efficiently to the wave field. As described above, dynamic synchronization[22–30] suggests that nonlinear auto-oscillators can be entrained to their mutual radiation field; they can synchronize to a common frequency with fixed phase differences. If that phase difference is such that each oscillator does work at a rate greater than it does without an incident field, i.e., if there is stimulated emission, the system will be a laser analog.[22]

Here we present the elements of one such acoustic system. A piezoelectric auto-oscillator design is described in the next section. A mathematical model is derived that shows potential for entrainment, synchronization and stimulated emission. Section III then presents a laboratory version of the model and makes quantitative observations of entrainment and stimulated emission under the influence of a prescribed applied acoustic wave. Finally, it is demonstrated (Sec. IV) that two or three piezoelectric auto-oscillators can synchronize autonomously (i.e, without an externally prescribed incident wave) to their mutual ultrasonic field. We observe stimulated emission and super radiance, and term the system a uaser.

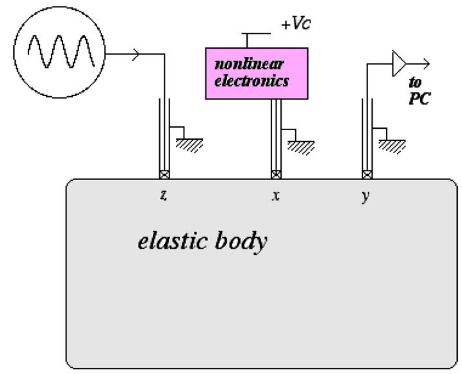

FIG. 1. (Color online) A reverberant elastodynamic body, large compared to a wavelength, is driven at position $\boldsymbol{x}$ by a piezoelectric transducer that is part of a circuit with an attracting limit cycle. The acoustic state of the body is monitored by a separate receiver at $\boldsymbol{y}$. An optional third transducer at $\boldsymbol{z}$ is driven by an applied harmonic signal at prescribed frequency and amplitude.

II. AUTO-OSCILLATION, ENTRAINMENT AND STIMULATED EMISSION

Figure 1 shows a piezoelectric transducer attached at $\boldsymbol{x}$ to an elastic body. Like most ultrasonic transducers it is reciprocal; it both radiates and receives ultrasound. It has an internal electronic impedance that is nominally capacitive, with additional small contributions from the mechanics. This transducer is part of a nonlinear electronic circuit (see Fig. 2) with a limit cycle at a frequency and amplitude that depends on circuit parameters. A second transducer at $\boldsymbol{y}$ monitors the acoustic state of the body. An optional third transducer at $\boldsymbol{z}$ applies a continuous harmonic wave at a prescribed frequency and amplitude. Additional limit-cycle oscillators (not pictured) may also be attached.

The auto-oscillator (also called a limit-cycle oscillator) will be influenced by its acoustic environment. Passive mechanical impedance felt by the piezoelectric element in con-

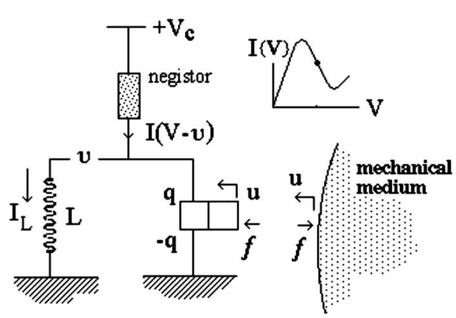

FIG. 2. Diagram for a piezoelectric auto-oscillator. The current I through the negistor depends on the potential across it in the pictured nonlinear fashion $I(V_c - \upsilon)$ characteristic of the device. The static supply voltage V_c is adjusted so that, at static equilibrium, $\upsilon = 0$ and the I(V) relation is at its inflection point. At this point current varies with υ like $\varepsilon C(1-\beta \upsilon^2)\upsilon$, with $\varepsilon > 0$. Further details of the negistor are described in Sec. III.

tact with the acoustic medium will manifest as electrical impedance and thus influence the auto-oscillation. Furthermore, any acoustic waves incident upon the piezoelectric element from elsewhere will generate additional currents in the circuit and so influence the oscillation. Thus we have an acoustic realization of systems studied in the literature,[22–30] consisting of forced and coupled limit-cycle oscillators. Of particular interest in this realization is that the interactions are by means of a complex wave bearing medium, and thus potentially irregular functions of time or frequency and position. An additional provocative question not found in that literature arises as well: whether the coupling enhances or diminishes the acoustic power flowing from the oscillator and thus whether a set of such oscillators may be a model for a laser.

In this section we propose an analytic model for this system and predict the frequency and rate of power emission at $\boldsymbol{x}$ in the absence of any source at $\boldsymbol{z}$. We also predict circumstances, with arguments similar to those of Refs. 23–28, under which the oscillator at $\boldsymbol{x}$ will be entrained by an applied field from $\boldsymbol{z}$. We further describe circumstances under which that entrainment corresponds, not only to locking, but also to stimulated emission. (The terms *synchronization*, *locking* and *entrainment* are defined for the present purposes in the footnote.[23]) This section establishes the possibilities then illustrated in Secs. III and IV. Sections III and IV can, however, be read independently.

The acoustic state of the body is monitored by a separate receiver at $\boldsymbol{y}$. An optional third transducer at $\boldsymbol{z}$ is driven by an applied harmonic signal at prescribed frequency and amplitude.

A. Linear transceiver

A passive linear reciprocal transducer may be represented in acoustics by a linear two-port network (see Ref. 32 for details), with potential drop $\boldsymbol{v}$ and charge $\boldsymbol{q}$ on one port and mechanical force $\boldsymbol{f}$ and mechanical displacement $\boldsymbol{u}$ on the other. These inputs and outputs are related by a matrix.

$$\begin{pmatrix} v \\ f \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{pmatrix} q \\ u \end{pmatrix}; \tag{1a}$$

$$\begin{pmatrix} q \\ u \end{pmatrix} = \begin{bmatrix} T_{22}/D & -T_{12}/D \\ -T_{21}/D & T_{11}/D \end{bmatrix} \begin{pmatrix} v \\ f \end{pmatrix}, \tag{1b}$$

where $D=T_{11}T_{22}-T_{12}T_{21}$ is $[T]$'s determinant. T_{11} may be interpreted as the inverse of the transducer's electric capacitance when clamped ($\boldsymbol{u}=0$). T_{22}/D may be interpreted as the capacitance C of the device when there is no force on it ($\boldsymbol{f}=0$). T_{22} is the shortcircuited mechanical stiffness of the device. Reciprocity demands $T_{12}=T_{21}$. All coefficients are in general complex functions of frequency; $T^{*}(\omega)=T(-\omega)$, and may in principle be measured.

The rate at which the force and electric potential are doing work on the transducer is

$$P=(vf)\times\begin{pmatrix} \partial q/\partial t \\ \partial u/\partial t \end{pmatrix} = (qu)[T]^{T}\times\begin{pmatrix} \partial q/\partial t \\ \partial u/\partial t \end{pmatrix}.$$

The symbol $\times$ represents, in the time domain, a simple multiplication; other juxtapositions represent time-domain convolutions. In the frequency domain these juxtapositions are simple multiplications. If $\boldsymbol{q}$ and $\boldsymbol{u}$ are harmonic: $\boldsymbol{q}=Q\exp(i\omega t)+\text{c.c}$: $\boldsymbol{u}=U\exp(i\omega t)+\text{c.c}$, we may conclude that the time average power flow into the transducer is

$$P=-i\omega(QU)[T(\omega)^{T}-T(\omega)^{*}]\begin{pmatrix} Q^{*} \\ U^{*} \end{pmatrix}$$

with $[T(\omega)]=\int\exp(-i\omega t)[T(t)]dt=[T(\omega)]^{T}$. A passive transducer must have $P\geqslant 0$. We therefore conclude that $[T(\omega)]$'s imaginary part must be positive (negative) semidefinite at positive (negative) frequency.

B. Linear piezoelectric transducer in a nonlinear electronic circuit

The transducer is inserted into the circuit of Fig. 2 in which the negistor has a nonlinear current voltage relation as pictured. At and near the inflection point in this curve the negistor has what is commonly termed negative differential electric resistance. When the negistor is placed in a standard *RLC* circuit, we expect oscillations with negative damping, oscillations that grow exponentially with time until such amplitude as the negative differential resistance $I(V)$ relation fails. The circuit can then be expected to settle into a limit cycle for which average resistance is zero. If the static bias voltage V_c is set at the inflection point, then fluctuations v of the voltage across the negistor correspond to fluctuating current $I(V_c-v)=I(V_c)+\varepsilon C(1-\beta v^2)v$. The slope at the inflection, $-\varepsilon C$, is the negative differential conductance. (The factor C has been inserted for later convenience.) Curvature at the inflection point is characterized by β. It governs the amplitude of the limit cycle.

Current conservation demands, where I_L is the current in the inductor,

$$I(V_c-v)=dq/dt+I_L.$$

This has a steady solution at $\boldsymbol{v}=\boldsymbol{q}=0$, $I_L=I(V_c)$. Recalling $dI_L/dt=\boldsymbol{v}/L$, differentiating with respect to time gives

$$-I'(V_c-v)dv/dt=d^2q/dt^2+v/L. \tag{2}$$

Substituting for d^2q/dt^2 by $(1b)$ gives

$$d^2v/dt^2-\varepsilon(1-3\beta v^2)dv/dt+(LC)^{-1}v=(T_{12}/T_{22})d^2f/dt^2. \tag{3}$$

This equation governs the fluctuations in voltage and could be useful if the force f were prescribed. In particular if $f=0$, corresponding to the transducer being mechanically free, (3) becomes a van der Pol equation.[33]

Equation (1b) implies a relation between f, u and v;

$$f=T_{21}/T_{11}v+D/T_{11}u. \tag{4}$$

So, if u is prescribed there is an alternate form

$$d^2v/dt^2 - \varepsilon(1-3\beta v^2)dv/dt + [(LC)^{-1} - \{T_{12}T_{21}/T_{22}T_{11}\}d^2/dt^2]v = (DT_{12}/T_{22}T_{11})d^2u/dt^2. \quad (5)$$

C. Spontaneous emission

Equations (3)–(5) are not complete unless we specify $\boldsymbol{u}$ or $\boldsymbol{f}$, or a relation between them. A case of primary interest is that of free radiation into an otherwise passive acoustic medium, for which: $u=-Gf$. (The minus sign is required due to the usual definition of elastodynamic Greens functions and to the opposite orientation of f and u in Fig. 2). Here G is the $\hat{n}\hat{n}$ component ($\hat{n}$ being the unit vector in the direction of $\boldsymbol{f}$ or $\boldsymbol{u}$) of the ultrasonic Greens' dyadic at position $\boldsymbol{x}$. [The Greens dyadic $G_{ij}(x,y,\omega)$ describes the complex amplitude of the linear elastodynamic displacement response of the solid in direction i at position $\boldsymbol{x}$ due to a unit harmonic force at $\boldsymbol{y}$ in direction j: so the quantity G used here is $n_j n_j G_{ij}(\boldsymbol{x},\boldsymbol{x},\omega)$]. Thus, from Eq. (4)

$$[1+DG/T_{11}]f = T_{21}/T_{11}v \quad (6)$$

and

$$d^2v/dt^2 - \varepsilon(1-3\beta v^2)dv/dt + [(LC)^{-1} - \{T_{12}T_{21}/T_{22}(T_{11}+DG)\}d^2/dt^2]v = 0. \quad (7)$$

Solutions of nonlinear dynamical equations such as Eq. (7) are difficult. This is especially so if the complicated time delays and temporal convolutions in T and G are fully expressed. For the present purposes we will presume that G may be evaluated at the frequency of chief interest, that G is only weakly dependent on frequency and can be replaced by a constant. The elements of $[T]$ also vary with frequency but generally in a much smoother fashion, so replacing them by constants is readily justified. Approximating Eq. (7) as a phase oscillator simplifies it further.[22–28] This requires that the oscillator be assumed to stay on or near its limit cycle, at a frequency Ω, with a slowly varying real amplitude and phase V, and ξ:

$$v = V(t)\exp(i\Omega t + i\xi(t)) + \text{c.c.} \quad (8)$$

On assuming $dV/dt \ll \Omega V$; $d\xi/dt \ll \Omega$, and $|\omega-\Omega| \ll \Omega$ we derive (this also requires an assumption of $\varepsilon \ll \Omega$.)

$$[2i\Omega dV/dt - \Omega^2 V + \omega^2 V - 2V\Omega d\xi/dt - i\varepsilon\Omega(1-6\beta V^2)V + i\mu V] = 0 \quad (9)$$

whose real and imaginary parts are

$$dV/dt - (1/2)\varepsilon(1-6\beta V^2)V + (1/2\Omega)\mu V = 0$$

$$d\xi/dt = (\omega^2-\Omega^2)/2\Omega \approx (\omega-\Omega), \quad (10)$$

where ω and μ are given by

$$\omega = (\mathrm{Re}[(LC)^{-1} + \{T_{12}T_{21}\Omega^2/T_{22}[T_{11}+DG]\}])^{1/2}$$

$$\mu = \mathrm{Im}[(LC)^{-1} + \{T_{12}T_{21}\Omega^2/T_{22}[T_{11}+DG]\}] \quad (11)$$

A steady solution implies $\Omega=\omega$, and $V=[(\varepsilon\Omega-\mu)/6\varepsilon\Omega\beta]^{1/2}$.

When the device is unattached to the solid body, as enforced mathematically by taking $f=0$, equivalently $G=\infty$, the frequency of the auto oscillation is the real part of $(LC)^{-1/2}$. When $G\neq\infty$, the frequency is changed by the acoustics to a value Ω that is a solution of the implicit equation [11(a) with $\Omega=\omega$.] In a reverberant body G is an irregular function of ω, so there may be multiple solutions.

The power radiated from the device into the mechanical medium is

$$\Pi = -f\times du/dt = f\times dG/dt\, f \quad (12)$$

which is, by Eq. (6),

$$\Pi = [1+DG/T_{11}]^{-1}T_{21}/T_{11}v \times dG/dt[1+DG/T_{11}]^{-1} \times T_{21}/T_{11}v$$
$$= -2\Omega\,\mathrm{Im}\,G(\Omega)|T_{11}+DG|^{-2}|T_{21}|^2V^2$$
$$= -2\Omega\,\mathrm{Im}\,G(\Omega)|T_{11}+DG|^{-2}|T_{21}|^2[(\varepsilon\Omega-\mu)/6\varepsilon\Omega\beta] \quad (13)$$

[The first equality in Eq. (13) arises from a time averaging.] This is the power of spontaneous emission.

Section III shows measurements on a system like that modeled here, with observations of the frequency of auto oscillation with and without $G=\infty$.

D. Entrainment

An incident wave field u^{inc} can modify the state of the nonlinear oscillator. The nonlinear oscillator augments the incident field with its own radiation $-Gf$, so instead of $u=-Gf$ we have $u=u^{\text{inc}}-Gf$. The equation governing the oscillator is still (3), but now in lieu of (6),

$$f = T_{21}/T_{11}v + D/T_{11}[u^{inc} - Gf] \quad (14)$$

so

$$d^2v/dt^2 - \varepsilon(1-3\beta v^2)dv/dt + [(LC)^{-1} - \{T_{12}T_{21}/T_{22}(T_{11}+DG)\}d^2/dt^2]v = (DT_{12}/T_{22}(T_{11}+DG))d^2u^{inc}/dt^2 \quad (15)$$

Equation (15) is identical to Eq. (7), except that now it has an external forcing term from the incident field. It is a van-der Pol oscillator with forcing. That such oscillators can be entrained to the frequency of the forcing is well known.[14,23–28] We take the incident field at x, u^{inc} to be of the form $U\exp(i\Omega t)+\text{c.c.}$ with independent parameter Ω and without loss of generality real positive U. The approximations used above now give

$$dV/dt - (1/2)\varepsilon(1-6\beta V^2)V + (1/2\Omega)\mu V = -(U\Omega/2)\mathrm{Im}\{DT_{12}\exp(-i\xi)/T_{22}(T_{11}+DG)\} \quad (16)$$

and

$$d\xi/dt = (\omega-\Omega) + (U/2V)\Omega\,\mathrm{Re}\{T_{12}\exp(-i\xi)/C \times (T_{11}+DG)\}. \quad (17)$$

This is similar to Adler's equation.[23,27,28]

For sufficiently small detuning,

$$|\omega - \Omega| < \Omega(U/2V)|T_{12}/C(T_{11} + DG)|, \tag{18}$$

the nonlinear differential equation [Eq. (17)] has stationary solutions at two distinct values of ξ. Each solution corresponds to entrainment of the oscillator to the incident field at frequency Ω. Only the solution with $\mathrm{Im}\{T_{12}\exp(-\mathrm{i}\xi)/C(T_{11}+DG)\}<0$ is stable.

E. Power flow and stimulated emission from an entrained oscillator

A general expression for the phase at stable entrainment is complicated, but in the absence of detuning ($\omega=\Omega$), it simplifies

$$\exp(-i\xi) = -i(DT_{12}/T_{22}(T_{11} + DG))^*/|DT_{12}/T_{22}(T_{11} + DG)|. \tag{19}$$

Examination of Eq. (16) shows that the stable solution for ξ corresponds to a positive right hand side, i.e., the stable phase at entrainment is such that the amplitude V of the oscillations around the inflection point is increased relative to its value in the absence of an incident field. Increased V corresponds to increased energy dissipation in the negistor. That conclusion is independent of the parameters $[T]$ of the two-port network, and independent of the degree of detuning. Thus this would appear to be *stimulated absorption*, and behaviors observed in the laboratory (see Sec. III) would appear to be not represented in this model.

That conclusion is incorrect. While the flow of energy out of the nonlinear circuit into the transducer is, apparently, decreased by the incident acoustic field, the flow of energy out of the transducer into the medium differs from that by dissipation in the transducer. Thus the model may yet describe stimulated emission, but only if the transducer is dissipative and only if power dissipation within the transducer is lessened by the presence of the incident field.

Thus we are led to ask about the radiation of acoustic energy into the mechanical medium. The power radiated from the device is $\Pi=-f\times du/dt=f\times\{-du^{\mathrm{inc}}/dt+dG/dt\, f\}$ which is [by Eq. (14)]

$$\begin{aligned}\Pi = {} & [T_{11} + DG]^{-1}(T_{21}v + Du^{inc}) \\ & \times dG/dt[T_{11} + DG]^{-1}(T_{21}v + Du^{inc}) \\ & - [T_{11} + DG]^{-1}(T_{21}v + Du^{inc}) \times \{du^{inc}/dt\}.\end{aligned} \tag{20}$$

The terms in v^2 are almost identical to the spontaneous emission rate, differing only in that V is now slightly different. The terms $u^{\mathrm{inc}\,2}$ are independent of V, and presumably due to passive losses on scattering off the dissipative parts of $[T]$. This presumption is supported by a short calculation that shows, for the case T=real, that the term in $u^{\mathrm{inc}\,2}$ vanishes. The cross terms, in $V\times U$, resemble stimulated emission and absorption. These terms are

$$\begin{aligned}\Pi_{UV} = {} & |T_{11} + DG|^{-2}[T_{21}v \times (dG/dt)Du^{inc} \\ & + Du^{inc} \times (dG/dt)T_{21}v - (T_{11} + DG)^* T_{21}v \times du^{inc}/dt]\end{aligned} \tag{21}$$

On time averaging over one cycle, we recover

$$\Pi_{UV} = 2\Omega UV|T_{11} + DG|^{-2}\,\mathrm{Im}[T_{21}^{*}\exp(-i\xi)(T_{11} + DG^*)].$$

A substitution of the expression (19) for ξ (valid in the absence of detuning) gives

$$\begin{aligned}\Pi_{UV} = {} & -2\Omega UV|T_{22}|/|T_{11} + DG||DT_{12}| \\ & \mathrm{Re}[T_{21}^{*2}(D/T_{22})^*[(T_{11} + DG^*)/(T_{11} + DG)^*]\end{aligned} \tag{22}$$

If T is real, this is manifestly negative. [We recall that $D/T_{22}=C$ is the free capacitance and its real part is presumably positive.] We therefore conclude, as in the paragraph following Eq. (18), that real T implies that the UV part of the power flow into the acoustic medium is negative, i.e, that an entrained oscillator absorbs energy from the field incident upon it.

If the elements of $[T]$ are complex the conclusion can differ. The expressions (20)–(22) are complicated, but permit some simplifications. For the case in which T is dominated by positive real parts of the elements on its diagonal, D and T_{11} are very nearly real; the ratio $[(T_{11}+DG^*)/(T_{11}+DG)^*]$ is unity; $(D/T_{22})\sim T_{11}$ is real and positive. We find Π_{UV} is positive if T_{21} is imaginary, *regardless of G*. Thus stimulated emission depends chiefly on the phase of T_{12}. If we suppose instead that T is dominated by the real part of T_{11}, with all other elements being complex and of equal small order, then $D\sim T_{11}T_{22}$ and

$$\begin{aligned}\Pi_{UV} = {} & -2\Omega UV|T_{22}|/|T_{11} + DG||DT_{12}| \\ & \mathrm{Re}[T_{21}^{*2}T_{11}(1 + T_{22}G^*)/(1 + T_{22}^{*}G^*)].\end{aligned} \tag{23}$$

In this case the sign of Π_{UV} depends on the phase of T_{12} and also, if the magnitude of $T_{22}G$ is of order unity or larger, on the phase of T_{22}.

We conclude that at least for some sets of parameters the model exhibits stimulated emission. For sufficiently weak detuning, and for transducers with lossy electromechanical coupling $[T]$, we expect an incident field to stimulate emission. These conclusions have been corroborated by direct numerical simulations (not presented) of Eq. (15). In particular we observe spontaneous emission and entrainment; we also see the predicted stimulated emission and its quantitative accord with theory.

III. LABORATORY STUDIES

We have investigated these predictions in the laboratory by constructing the "negistor," or "lambda diode,"[34] illustrated in Figs. 2 and 3, and incorporating it into the circuit of Figs. 1 and 2. We find that the circuit auto-oscillates in a periodic nearly harmonic limit cycle whose frequency is tunable by varying inductance L or by adding additional capacitance. (The parameter ε describing the strength of the negative resistance is also tunable, by adding additional resistance.) Here, and in all cases below, the spectral width of the auto-oscillation is finer than our precision of 1–10 Hz. Linewidth may be governed by background noise as is the Schawlow–Townes theory for the linewidth of a laser[35] or a Larsen feedback circuit.[6] In the following subsection, the circuit of Fig. 3 is shown to oscillate in a limit cycle with a frequency ω that depends, as predicted in Eq. (11), on the complex mechanical compliance represented by G. The next

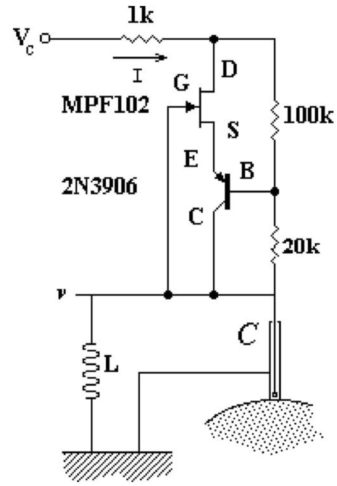

FIG. 3. A lambda diode, or negistor, is constructed from two transistors. It has a current voltage relation as shown in the inset to Fig. 2. When it is incorporated into a circuit with an inductor $L(\sim 220\ \mu\text{H})$ and a capacitance C as provided by the piezoelectric transducer and its cables (~200 pf), the circuit auto-oscillates at a frequency of about $\omega = 1/\sqrt{LC}$.

subsection demonstrates that the circuit will be pulled or entrained by an applied field depending on the degree of detuning and the amplitude of the applied field, as predicted in Eq. (18). Finally it is shown in III C that the applied field can enhance or diminish the acoustic power from the auto-oscillator, as predicted in Sec. II D.

A. Free acoustic oscillation, spontaneous emission

When the piezoelectric transducer with its nonlinear electronics is attached to an elastic body, the frequency of oscillation shifts, as shown in Fig. 4. Frequency changes can be different depending on the position of attachment, the material, the size of the solid body, or the presence of oil couplant. Frequencies always increase when the transducer is placed on the body, reflecting an increase in effective stiffness. No change in frequency is attendant upon additional grounding of the transducer case, so changes may be ascribed to the mechanics, not the electronics. Changes are small compared to natural frequency of the order of 500 kHz.

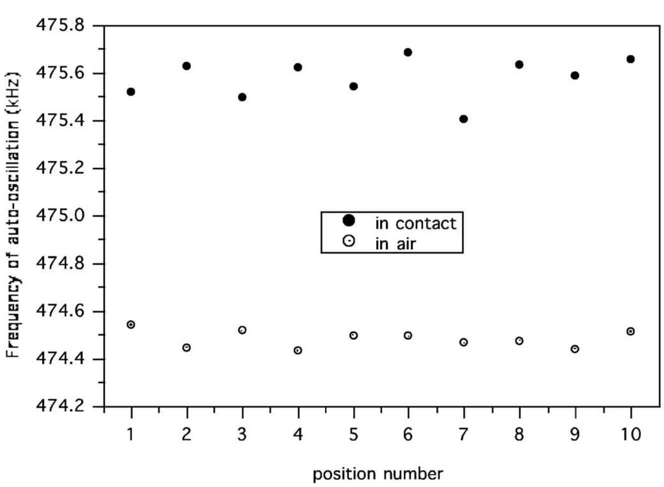

FIG. 4. Frequency of auto oscillation varies by about 1 kHz as the transducer is alternately in contact (filled circles) and out of contact (open circles) with the aluminum block. Slight variations in the frequency of the noncontact case are ascribed to stray capacitances due to the operator's hands. Variations in frequency among the filled circles are ascribed to variations in coupling strength (reattachment of a transducer is notoriously resistant to precise reproducibility) or to variations in local Greens function.

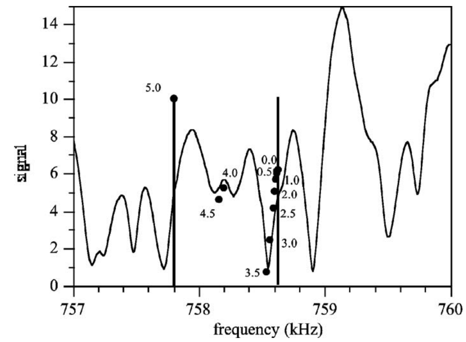

FIG. 5. The nonlinear oscillator's frequency (isolated points) is pulled, monotonically and discontinuously, as the signal strength from the generator at $\boldsymbol{y}$ is increased from 0 to 5 V in 0.5 V steps. The continuous curve is the transfer function $|h(\omega)|$ between monitor and generator. The bold vertical lines mark the positions of the spectral lines (one at 757.79 due to the generator, another at 758.63 due to the uninfluenced auto-oscillator) at vanishing voltage from the generator. The isolated filled circles are labeled with the corresponding values of g in volts; their positions indicate the pulled frequency and the signal strength from the auto-oscillator.

Theoretical frequency is given by Eq. (11). From the observed magnitude of the frequency changes in Fig. 4 as G is alternated between ∞ and the G of the elastic body in different places, and based upon statistically identical data from steel where G is smaller than it is in aluminum, we estimate Re $(T_{12}T_{21}/T_{22}T_{11})$ to be of the order 0.004, and DG to be less than T_{11}. It is worth noting that in a system with high modal overlap (meaning level spacings are much less than absorption widths) such as the one used in generating Fig. 4, G has fluctuations from place to place or frequency to frequency that are small; $|\delta G|/|G| \ll 1$.[36] G is very nearly equal to the value it would take in an infinite half space. For this reason the frequency of auto-oscillation ought to depend only weakly on position $\boldsymbol{x}$.

B. Entrainment to an applied field

Entrainment is studied by adding a prescribed incident field through the transducer at point z in Fig. 1. Conditions are sought under which the nonlinear electronic circuit adopts the frequency of the applied field. Figure 5 shows (continuous curve) a short section of the spectrum of the ultrasonic transfer function between the generator and the monitor, $|h_{zy}|$ on a 70 mm aluminum cube with enhanced dissipation. (On ignoring scattering by the transducers one can calculate this transfer function in terms of other quantities that have been defined here: $h_{zy} \approx T_{12}G_{zy}T_{21}/(1 + T_{22}G_{zz})(1+T_{22}G_{yy}) \sim T_{12}^2 G_{zy}$.) The bold vertical lines indicate the frequency of the generator at z (757.79 kHz) and that of the undisturbed oscillator in contact with the solid at $\boldsymbol{x}$ (758.63 kHz). The several isolated points indicate the frequency and rms amplitude received at y, due to the oscillator, at each of 11 equally spaced generator amplitudes from 0.0 to 5.0 V. It may be seen that the oscillator frequency is *pulled*[23] towards that of the generator, and that the degree of

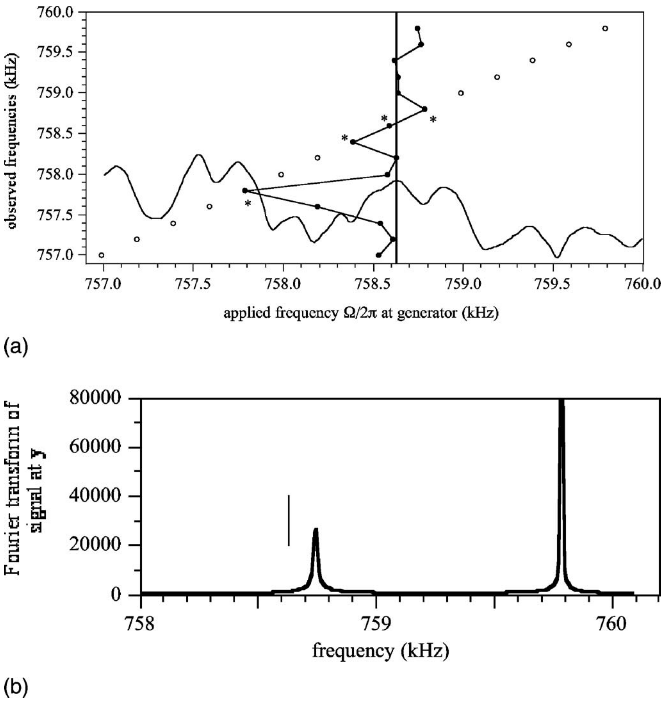

FIG. 6. (a) A study of auto-oscillator frequency *versus* the frequency of an entraining field with a fixed harmonic source amplitude $g=5$ V. The natural frequency (i.e., in the absence of an applied field) of the auto-oscillator at $\boldsymbol{x}$ is 759.63 kHz, as indicated by the vertical line. The irregular smooth curve shows the (scaled to fit) transfer function $|h_{xz}(\omega)|$ between the source of the entraining field and the position of the auto-oscillator. Open circles indicate the frequency of the entraining field as specified by the operator. Closed circles indicate the frequency of the auto-oscillator as it is pulled towards that of the entraining field. The structure of the entrainment is complex, and the pulling is nonmonotonic and discontinuous, influenced as it is by the complicated function $h(\omega)$. Entrainment occurs for the points labeled *. (b) An example of the spectra (of the signals received at $\boldsymbol{y}$) used to construct Fig. 6(a). The strong peak at 759.79 kHz corresponds to a direct signal from the generator at z to the monitor at $\boldsymbol{y}$. The weaker peak at 758.73 kHz indicates the frequency chosen by the auto-oscillator. It has been pulled, from its original value of 758.62 indicated by the light vertical line segment. Were the auto-oscillator to be entrained we would see only the peak at 759.79. This plot corresponds to the uppermost points in Fig. 6(a).

pulling is monotonic in generator strength. This behavior is familiar in the Adler equation from the literature on entrainment of a single auto-oscillator.[28] Less familiar are the occasional discontinuities. The two discontinuities,between 3.5 and 4.0 V, and between 4.5 and 5.0 V where locking ensues, appear to correspond to features in the transfer function h of the solid body. They are related to the random reverberant nature of the wave propagation and are not found in the standard (e.g., Refs. 23 and 24) model with frequency-independent coupling.

Figure 6(a) shows the behavior of entrainment as the frequency of the applied continuous harmonic signal from the generator is varied, at a fixed amplitude of 5.0 V. Frequency is varied in 0.2 kHz steps from 756.99 to 759.79 kHz. The natural frequency of the auto oscillator without applied field is indicated by the vertical line at 758.63 kHz. In the absence of full entrainment the auto-oscillator frequency is pulled towards that of the generator. The auto-oscillator is entrained to the applied frequency if the applied frequency is close to the natural frequency, and if the signal from the generator at z as received at the auto-

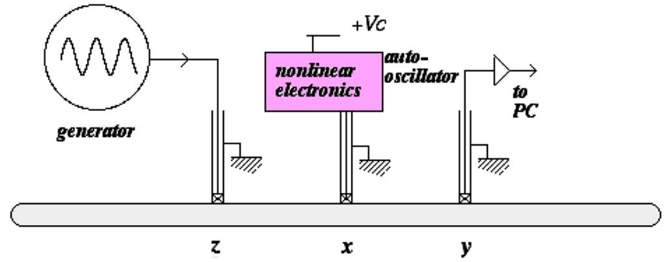

FIG. 7. (Color online) The system is attached to an aluminum rod of 15 cm length and 3.16 mm diameter.

oscillator at $\boldsymbol{x}$ is strong enough, i.e., if the transfer function is large. Entrainment is a complicated nonmonotonic function of applied frequency. Figure 6(b) shows the data used to determine the uppermost points in 6(a), the spectrum at $\boldsymbol{y}$ for the case $\Omega/2\pi=759.79$ kHz at which the oscillator was pulled to $\omega/2\pi=758.73$ kHz.

C. Stimulated emission and absorption

We also study the energy in the system as a function of the amplitude of the applied field, and do so with focus on the case of no de-tuning, i.e., for $\Omega=\omega$. Being unable to measure the power inflow at $\boldsymbol{x}$, we instead measured the signal strength at $\boldsymbol{y}$ and interpreted it as a measure of the (square root of the) energy in the cavity. This is an imperfect measure of acoustic energy unless the system has low modal overlap and the frequency of interest is near one of the modes, so that only one natural mode is excited. Thus we chose a thin rod with low modal density and modest absorption as pictured in Fig. 7.

We also recognize that the total energy in the rod is proportional to the sum of the power flows from the nonlinear circuit at $\boldsymbol{x}$ and the generator at z. The energy radiated into the rod may be decomposed in two terms, each equal to the time average of the dynamic force at a transducer times the material velocity at the same point. This quantity at $\boldsymbol{x}$, it was shown above, has a term in V^2 and a term in UV, where U is the field incident from the prescribed forcing at z and proportional to the continuous wave signal input strength "g" from the generator. The work done at z, however, includes not only a term in g^2, but also a term in gV due to the field incident from $\boldsymbol{x}$. Thus the ultrasonic energy radiated into the body includes not only the intuitive g^2 and V^2 terms, and the gV term described in Sec. II D, but another term scaling like gV that was not discussed there. Analysis shows that the excess stimulated emission of energy at z is of the same order as that at $\boldsymbol{x}$, but with a phase depending on the phase of G_{zx}. It is not necessarily positive, even if Π_{UV} is positive. Nevertheless, for the case of low modal overlap, the two terms can be shown to have the same sign and the total energy in the cavity can be a proxy for the power flow from the nonlinear circuit.

Figure 8 shows the rms of the continuous harmonic signal observed at the monitor at $\boldsymbol{y}$ for various levels g of the applied input at z. Figure 8(b) shows the difference of the squares of the curves in 8(a), thus corresponding to the energy of spontaneous emission (the value at $g=0$) plus the UV part linear in g that corresponds to stimulated emission.

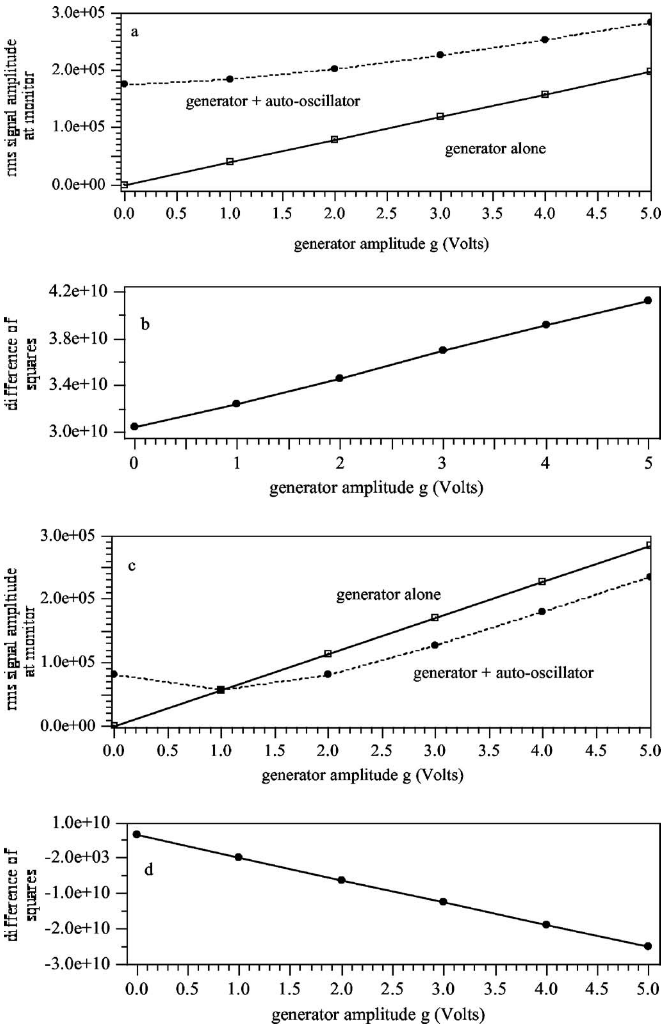

FIG. 8. Evidence of stimulated emission and absorption in the rod of Fig. 7. The rms signal amplitude at $\boldsymbol{y}$ is plotted for two (a, and c) transducer positions $\boldsymbol{x}$ against the amplitude of the signal from the generator which drives the transducer at z. Filled circles correspond to the case in which the auto oscillator is powered; empty squares to unpowered. The difference of their squares is also plotted (b,d), indicating a coherent interference, i.e, stimulated emission (a,b) and absorption (c,d).

Stimulated emission is apparent in the positive slope in Fig. 8(b); power output is greater than the sum of the power outputs from the nonlinear circuit and the generator when they operate alone. Figures 8(c) and 8(d) show the stimulated absorption observed with a different transducer position $\boldsymbol{x}$.

IV. UASING

On replacing the generator-driven transducer at z with one or more additional van der Pol auto-oscillating transducer circuits, as in Fig. 9, we have what may be termed a "uaser" (Ultrasound amplification by stimulated emission of radiation[21]), an acoustic analog to a laser. Of special interest are circumstances under which all oscillators synchronize to their mutual radiation field. One imagines that, as in the Kuramoto model,[23–27] one would describe a large number of such synchronizing auto-oscillators as experiencing a phase transition. This differs from the directed entrainment discussed in Sec. III in which an oscillator locks to a prescribed applied field. The system here is similar to that encountered

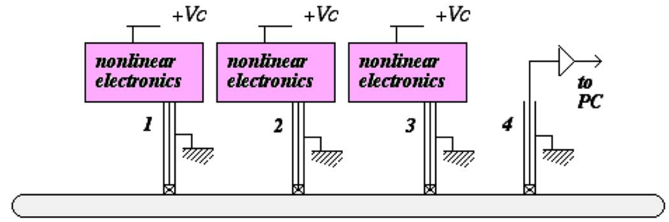

FIG. 9. (Color online) Two or three autonomous nonlinear piezoelectric oscillators are placed in contact with the aluminum rod of Fig. 7. The acoustic state is monitored with a passive detector.

when two or more Josephson junctions couple and synchronize through their microwave radiation field, a system noted elsewhere as analogous to a laser.[20] The data presented here are not meant to fully elucidate the properties of such systems. Rather these illustrations are presented to provoke imagination and suggest further studies. A thorough study awaits development of a theory to inform such experiments.

Figure 10 shows the behavior of the system of three auto-oscillators (labeled 1, 2 and 3) monitored by a receiver (labeled 4). For reference, we plot the three transfer functions h_{14}, h_{24}, and h_{34} between the auto-oscillators 1, 2 and 3, and the monitor at 4. Bold vertical lines indicate the rms amplitudes Uj received at 4 from each individual auto-oscillator when other oscillators are turned off, of each pair, Uij when only one is off, and U123 when all three are on. It is apparent that any pair of oscillators synchronize to each other, and that all three oscillators synchronize to each other also. Frequency of synchronization is not always between the natural frequencies of the individual oscillators. This synchronization is similar to that described elsewhere (see Refs. 23–26, especially Ref. 27, Chapters 10 and 11). We may, however, also examine issues of ultrasonic power emission, stimulated emission and super radiance.

It behooves us to renormalize the amplitudes when comparing amplitudes emitted by an oscillator at two different frequencies, as the transfer function of the structure varies with frequency. An amplitude as measured by a receiver at position 4 due to emissions from oscillator j at frequency f_j $(j=1,2,3)$ may be renormalized for comparison with the

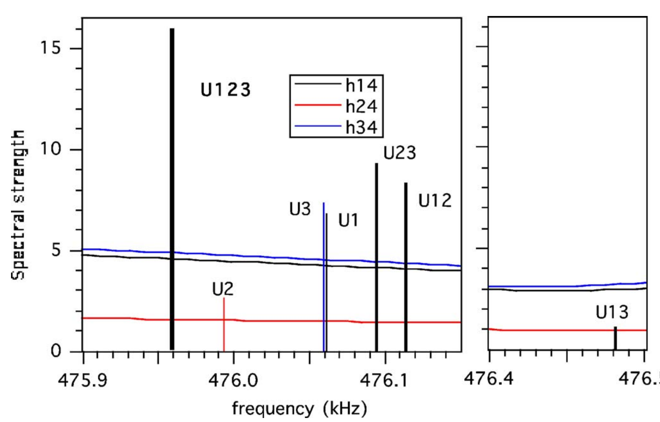

FIG. 10. (Color online) Three auto-oscillators are monitored by a single receiver at point 4. The rms signal strengths (rescaled to fit on the plot) are indicated by the vertical lines. Amplitudes of single auto-oscillators (thin vertical lines) are compared to those when auto-oscillators are taken in pairs (wider lines), and when all three auto-oscillators are powered (widest line).

same oscillator's contribution at another frequency f, by replacing the rms amplitude Uj with Uj times $h_{j4}(f)/h_{j4}(f_j)$. We compare the amplitude produced by the pair of oscillators 1 and 2 (labeled U12 in the figure) with their emissions U1 and U2 when isolated. The normalized amplitudes U1 and U2 of individual oscillators 1 and 2 (6.53 and 2.43 when referred to frequency f=476.114) add to approximate the amplitude of that pair together (U12=8.27). The normalized amplitudes of oscillators 2 and 3 (2.45 and 7.15 when referred to frequency 476.095) add to approximate the amplitude of that pair together (9.20). These pairs both add constructively. Oscillators 1 and 3 have normalized amplitudes (5.19 and 4.73 when referred to frequency 476.482) that roughly subtract to approximate the amplitude (1.05) of that pair together. Oscillators 1, 2 and 3 (normalized amplitudes 7.31, 2.66, and 7.88, respectively, when referred to frequency 475.959) add to approximate the amplitude of all three together (15.93). These comparisons show that acoustic energy is generated coherently, and at more than twice the rate that would be obtained in the absence of feedback. The system is super-radiant.[36]

V. SUMMARY

It has been shown that nonlinear van-der Pol-like piezoelectric oscillators can be configured to exhibit the key behaviors required of sets of neighboring continuously pumped "atoms" in a classical analog for a laser. These include frequency locking, stimulated and spontaneous emission, stimulated absorption, and super-radiance. We conjecture that the principles illustrated here may find direct application in the construction of new kinds of acoustic generators, and indirect application in scale-model emulation of laser dynamics, in particular in research on random[31] and chaotic[37] and photonic crystal lasers[38,39] and arrays of Josephson junctions.[22,30]

ACKNOWLEDGMENTS

This work was supported by the NSF, Grant No. CMS 05-28096. A.Y. acknowledges support from University of Missouri-Rolla.

[1]K. Popp and P. Stelter, "Stick-slip vibrations chaos," Philos. Trans. R. Soc. London **332**, pp. 89–105 (1990); S. N. Patek, "Spiny lobsters stick and slip to make sound," Nature (London) **411**, 153–154 (2001).

[2]S. Adach, "Principles of sound production in wind instruments," Acoust. Sci. & Tech. **25**, 400–405 (2004); T. D. Mast and A. D. Pierce, "Describing-function theory for flow excitation of resonators," J. Acoust. Soc. Am. **97**, 163–172 (1995).

[3]P. van Dijk and H. P. Wit, "Amplitude and frequency fluctuations of spontaneous otoacoustic emissions," J. Acoust. Soc. Am. **88**, 1779–1793 (1990).

[4]G. W. Swift, "Thermoacoustic engines," J. Acoust. Soc. Am. **84**, 1145–1180 (1988).

[5]A. Larsen, "Ein akustischer Wechselstromerzeuger mit regulierbare Periodenzahl für schwache Ströme," Elektrotechnische Zeitschrift, ("An acoustic alternating-current generator for weak currents, with adjustable frequency") Elektrotech. Z. **32**, 284–285 (1911); A. Kjerbye Nielsen, "'Larsen-effekten' og den fØrste elektriske tonegenerator baseret herpa," ("The 'Larsen effect' and the first electrical pure-tone generator based on this effect".) Teletek. (Engl. ed.) **3**, 140–481 (1984); D. Barbaro, "Self starting acoustic oscillations in closed spaces," Alta Freq. **27**, 472–85 (1958).

[6]R. L. Weaver and O. I. Lobkis, "On the line width of the ultrasonic Larsen effect in a reverberant body," J. Acoust. Soc. Am. **120**, 102–109 (2006).

[7]A. Furduev, "Acoustic monitoring of the sea meduim varaibility: Experimental testing of new methods," Acoust. Phys. **47**, 361–268 (2001).

[8]J. M. Kates, "Constrained adaptation for feedback cancellation in hearing aids," J. Acoust. Soc. Am. **106**, 1010–1019 (1999).

[9]S. M. Kim, S. J. Elliott, and M. J. Brennan, "Decentralized control for multichannel active vibration isolation," IEEE Trans. Control Syst. Technol. **9**, 93–100 (2001).

[10]P. S. Spoor and G. W. Swift, "The Huygens entrainment phenomenon and thermoacoustic engines," J. Acoust. Soc. Am. **108**, 588–599 (2000); P. S. Spoor and G. W. Swift, "Mode-locking of acoustic resonators and its application to vibration cancellation in acoustic heat engines," J. Acoust. Soc. Am. **106**, 1353–l362 (1999).

[11]I. I. Blekhman, *Synchronization in Science and Technology* (ASME Press, New York 1988); B. Bennett, M. F. Schatz, H. Rockwood, and K. Wiesenfeld, "Huygens' clocks," Proc. R. Soc. London, Ser. A **458**, 563–579 (2002).

[12]J. Pantaleone, "Synchronization of metronomes," Am. J. Phys. **70**, 992–1000 (2002).

[13]M. Borenstein and W. E. Lamb, "Classical laser." Phys. Rev. A **5**, 1298 (1972).

[14]M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison–Wesley, Reading, MA, 1974).

[15]B. Fain and P. W. Milonni, "Classical stimulated-emission," J. Opt. Soc. Am. B **4**, 78–85 (1987).

[16]Y. A. Kobelev, L. A. Ostrovsky, and I. A. Soustova, "Nonlinear model of autophasing of classical oscillators," Zh. Eksp. Teor. Fiz. **99**, 470–480 (1991).

[17]V. V. Bredikhin, Yu. A. Kobelev, and N. I. Vasilinenko, "Autophasing of free volume oscillations of air cavities in water," J. Acoust. Soc. Am. **103**, 1775–1786 (1998).

[18]S. T. Zavtrak, "Acoustical laser with mechanical pumping," J. Acoust. Soc. Am. **99**, 730–733 (1996).

[19]I. V. Volkov, S. T. Zavtrak, and I. S. Kuten, "Theory of sound amplification by stimulated emission of radiation with consideration for coagulation," Phys. Rev. E **56**, 1097 (1997).

[20]B. G. Danly and R. J. Temkin, "Generalized nonlinear harmonic gyrotron theory," Phys. Fluids **29**, 561–567 (1986).

[21]A. Yamilov, R. Weaver, and O. Lobkis, *UASER: Ultrasound Amplification by Stimulated Emission of Radiation*, Photonics Spectra (August 2006).

[22]P. Barbara, A. B. Cawthorne, S. V. Shitov, and C. J. Lobb, "Stimulated emission and amplification in Josephson Junction arrays," Phys. Rev. Lett. **82**, 1963–1966 (1999).

[23]A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear sciences*, (Cambridge University Press, Cambridge 2001). The terms *synchronization, locking* and *entrainment* are used in the literature in various ways. In the current paper *entrainment* is taken to describe the state of a limit-cycle oscillator when its frequency conforms precisely to that of an external forcing. We choose the term *locking* for the same case, but to emphasize the fixed phase relation. The term *synchronization* is used here to describe the autonomous entrainment of a set of auto-oscillators to their mutual radiation field in the absence of a prescribed forcing. A fourth term, *pulling*, describes change—but without full conformation—of the frequency of a nonlinear oscillator under the influence of a continuous harmonic force.

[24]S. H. Strogatz, "From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators," Physica D **143**, 1–20 (2000).

[25]H. Daido, "Quasi-entrainment and slow relaxation in a population of oscillators with random and frustrated interactions," Phys. Rev. Lett. **68**, 1073–1076 (1992).

[26]P. C. Matthews, R. E. Mirollo, and S. H. Strogatz, "Dynamics of a large system of coupled nonlinear oscillators," Physica D **52**, 293–331 (1991).

[27]P. S. Landa, *Regular and Chaotic Oscillations* (Springer New York, 2001).

[28]R. Adler, "A study of locking phenomena in oscillators," Proc. IEEE **61**, 1380–1385 (1973).

[29]Y. Kuramoto, *Chemical Oscillations and Waves* (Springer-Verlag, New York, 1984).

[30]P. Hadley, M. R. Beasly, and K. Wiesenfeld, "Phase locking of Josephson-junction series arrays," Phys. Rev. B **38**, 8712–8719 (1988); K. Wiesenfeld, K. P. Colet, and S. H. Strogatz, "Synchronization transitions in a disordered Josephson series array," Phys. Rev. Lett. **76**, 404–407 (1996).

[31]D. S. Wiersma, M. P. VanAlbada, and A Lagendijk, "Random laser," Nature (London) **373**, 203–204 (1995); C. Vanneste, and P. Sebbah, "Selective excitation of localized modes in active random media," Phys. Rev.

Lett. **87**, 183903 (2001).
[32]G. S. Kino, *Acoustic Waves* (Prentice–Hall, Englewood Cliffs. NJ, 1987).
[33]M. Lakshmanan and K. Murali, *Chaos in Nonlinear Oscillators. Controlling and Synchronization* (World Scientific, Singapore, 1996).
[34]Ramon Vargas-Patron, personal communication; see also (last seen online 7/2007) http://cidtel.inictel.gob.pe/cidtel/contenido/Publicaciones/rvargas/ORANRD.pdf
[35]A. L. Schawlow and C. H. Townes, "Infrared and optical masers," Phys. Rev. **112**, 1940 (1958).
[36]R. L. Weaver, "Wave chaos in elastodynamics, pp. l41–l86 in *Waves and Imaging through Complex Media*, edited by P. Sebbah, *Proceedings of the International Physics School on Waves and Imaging through Complex Media*, Cargese, France (Kluwer, Dordrecht, 2001).
[37]We use the term in the sense of R. H. Dicke, Phys. Rev. **43**, 102 (1954). "For want of a better term, a gas which is radiating strongly because of coherence will be called 'super-radiant'" to refer to emission that scales with the number N of emitters faster than the first power.
[38]V. A. Podolskiy, E. E. Narimanov, W. Fang, and H. Cao, "Chaotic microlasers based on dynamical localization," Proc. Natl. Acad. Sci. U.S.A. **101**, 10498–10500 (2004).
[39]S. Strauf, K. Hennessy, M. T. Rakher, Y. S. Choi, A. Badolato, L. C. Andreani, E. L. Hu, P. M. Petroff, and D. Bouwmeester, "Self-tuned quantum dot gain in photonic crystal lasers," Phys. Rev. Lett. **96**, 127404 (2006).