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★**Dynamic equations on time scales. (English summary)**

An introduction with applications.

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FEATURED REVIEW.

For a long time there has been an interest in studying and understanding the similarities and analogies between the theories of differential equations and forward difference equations with constant step size. Discrete Sturm comparison and separation theorems were presented by T. Fort in 1948 [*Finite Differences and Difference Equations in the Real Domain*, Oxford, at the Clarendon Press, 1948; MR0024567 (9,514a)]. In 1978, P. Hartman [Trans. Amer. Math. Soc. **246** (1978), 1–30; MR0515528 (80a:39004)] developed the theory of disconjugacy of nth-order forward difference equations with constant step size in complete analogy to the rich theory of disconjugacy for nth-order ordinary differential equations. The results at that time were quite striking; moreover, completely new discrete methods were developed. Hartman's paper initiated considerable activity in the study of the theory of forward difference equations as an analogue to the theory of ordinary differential equations.

S. Hilger pushed these studies to a new level in 1988 with his Ph.D. thesis ["Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten", Univ. Würzburg, Würzburg, 1988; Zbl 0695.34001]. He defined a time scale and a delta derivative of a function defined on that time scale. He then initiated the calculus on time scales. If the time scale is an interval, the calculus reduces to the classical calculus; if the time scale is discrete, the calculus reduces to the calculus of finite differences. Thus, an initial motivation to develop and study calculus on time scales was to provide the unification of continuous and discrete calculus. Hilger's work has led to considerable activity as the theory of dynamic equations on time scales is currently undergoing rapid development.

The monograph under review comes at an excellent time in the rapid development of dynamic equations on time scales. Both authors are authorities in this field of study and they have produced an excellent introduction to it. Much of the material is accessible to upper-level undergraduate mathematics majors, and yet, the results and the techniques are pertinent to active researchers in the area.

In Chapter 1, the authors develop the time scales calculus. Polynomial calculus and even Taylor formulas are developed. The polynomials are developed iteratively; zero-order polynomials are spanned by the 1 function and the higher-order polynomials are obtained by integrating iteratively. Integration is currently a difficult concept with which to work. The integral is defined in the context of antiderivatives and a fundamental theorem; in particular, a Riemann-type integral has yet to be produced. The chain rule is interesting. Once one has become comfortable with analysis on time scales, the chain rule seems to be quite natural. It was obtained by C. D. Ahlbrandt, M. Bohner and J. R. Ridenhour [J. Math. Anal. Appl. **250** (2000), no. 2, 561–578; MR1786081 (2001i:34017)];

I believe that before that time, it was the common perception that a chain rule was not valid for difference calculus. We did not understand that there were, in fact, two time scales for which to account.

In Chapter 2, the authors present Hilger's complex plane [Dynam. Systems Appl. **8** (1999), no. 3-4, 471–488; MR1722974 (2001g:39015)] and they develop the exponential and other transcendental functions. It is quite interesting that while time scales polynomial calculus is not elegant, the development of the other elementary functions is quite elegant.

Chapters 3 and 4 read very similarly to a textbook for a first course in ordinary differential equations. Here the authors study reduction of order, variation of parameters, Euler-Cauchy equations, the annihilator method, transform methods, the Riccati equation, self-adjointness, and more. With respect to the Laplace transform method, the z-transform has served as the discrete transform method for forward equations with constant step size. The z-transform method is motivated and developed by a moment generating function method and can be obtained from the method presented in this monograph with a change of variable. The advantage of the method given here is that one calculus gives both the continuous and the discrete transform. It becomes very striking in these chapters as to how well the time scales calculus does unify the continuous and discrete calculus.

In Chapter 5, the authors introduce linear systems and higher-order scalar equations; they extend classical inequalities such as the Gronwall, Jensen and Opial inequalities in Chapter 6, and they introduce linear symplectic dynamic equations in Chapter 7. In the final chapter, Chapter 8, they introduce several concepts that have yet to attract sufficient attention, and the concepts indicate clearly that there is considerable work to do to develop the theory of dynamic equations on time scales.

The monograph stresses two features of time scales calculus, unification and extension. Only unification has been addressed above. With respect to unification, the book is very well written. The topics are well chosen to illustrate the unification features in a striking way. While the results are elegant, many methods of proof are not. The authors provide a wide variety of technical arguments so that the interested reader can readily begin to contribute new mathematics. There is still considerable work to be accomplished even in the area of unification.

Work is only beginning in the area of extension and there is potential for solid contributions to applications of mathematics. First, and most simply, there has already been solid contribution to the study of discrete calculus. For example, the chain rule for time scales is proved. This leads to immediate applications such as change of variable formulas that are new as discrete formulas. Second, there is potential for solid contributions to modeling. Is a model always discrete, is a model always continuous, or can there be gray areas in modeling? The authors give some examples, and further exploration should lead to fruitful development. Third, there is potential for solid contributions to computation and numerical methods. Even in the case when the domain is a finite set of points and especially when the step size is not constant, computations in numerical methods do not take full advantage of a calculus on time scales.

Reviewed by *P. W. Eloe*