23. Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \ x_1, x_2 \in \mathbb{R} \right\}$ and define addition "+" by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ whenever $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in V$ and scalar multiplication "$\cdot$" by $c \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$ whenever $c \in \mathbb{R}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V$. Show in detail that $(V, +, \cdot)$ is a real vector space.

24. Let $n \in \mathbb{N}$ and let V be the set of polynomials $p : \mathbb{R} \to \mathbb{R}$ with degree not exceeding n and define addition "+" by $(p+q)(x) = p(x) + q(x)$ for all $x \in \mathbb{R}$ whenever $p, q \in V$ and scalar multiplication "$\cdot$" by $(c \cdot p)(x) = cp(x)$ for all $x \in \mathbb{R}$ whenever $c \in \mathbb{R}$ and $p \in V$. Show in detail that $(V, +, \cdot)$ is a real vector space.

25. Determine whether the following $(V, +, \cdot)$ are real vector spaces and justify your claims.

(a) $V = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} : \ x \in \mathbb{R} \right\}$, $\begin{bmatrix} 1 \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ x+y \end{bmatrix}$, $c \cdot \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ cx \end{bmatrix}$;

(b) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \ x_1, x_2 \in \mathbb{R} \right\}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2 + x_2 \\ y_1 + 2 + y_2 \end{bmatrix}$, $c \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$;

(c) $V = (0, \infty)$, $x + y = xy$, $c \cdot x = x^c$;

(d) V the set of all invertible 2×2-matrices, with usual matrix addition and multiplication of a matrix by a scalar;

(e) V the set of all non-invertible 2×2-matrices, with usual matrix addition and multiplication of a matrix by a scalar.

26. Which of the following sets are subspaces of $\mathbb{R}^3$? Again, justify your claims.

(a) The set of vectors in $\mathbb{R}^3$ with first component 0;

(b) The set of vectors in $\mathbb{R}^3$ with last component 4;

(c) The set of vectors in $\mathbb{R}^3$ whose components multiplied together gives zero;

(d) The set of vectors in $\mathbb{R}^3$ whose first two components are the same;

(e) The set of all linear combinations of the two vectors $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$;

(f) The set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ whose components satisfy $4a - b + 2c = 0$;

(g) The set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ whose components satisfy $4a - b + 2c - 4 = 0$.

27. Let V be the vector space consisting of all 3×3-matrices (with usual matrix addition and multiplication of a matrix by a scalar). Find the smallest subspace which contains all symmetric matrices and all lower triangular matrices. What is the largest subspace which is contained in both of these subspaces?

28. Let V be a vector space and let U_1 and U_2 be subspaces. Prove that $U_1 \cap U_2$ is also a subspace of V. How about $U_1 \cup U_2$?

29. Find the column space and the row space of the following matrices:

    (a) $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$;

    (b) $\begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix}$;

    (c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

    (d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.