30. For each of the following matrices A, find $\mathcal{N}(A)$ and $\mathcal{R}(A^T)$. Draw a picture featuring these two spaces.

(a) $A = \begin{bmatrix} 2 & 1 \end{bmatrix}$; (b) $A = \begin{bmatrix} -2 & -1 \end{bmatrix}$; (c) $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$;

(d) $A = \begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix}$; (e) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; (f) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$;

(g) $A = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$; (h) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 3 \end{bmatrix}$; (i) $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 3 & 3 & 2 \end{bmatrix}$.

31. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) Find the smallest subspace U_1 of R^3 that contains v_1.

(b) Find the smallest subspace U_2 of R^3 that contains v_2.

(c) Find the sum U of these two subspaces U_1 and U_2, that is, the set of all possible combinations $x + y$, where $x \in U_1$ and $y \in U_2$.

(d) Finally find a subspace U_3 that satisfies $U + U_3 = \mathbb{R}^3$ and $U \cap U_3 = \{0\}$.

32. Find the sum of $\mathcal{N}(A)$ and $\mathcal{R}(A^T)$ for each of the matrices A from Problem 30.

33. We call a matrix P idempotent if $P^2 = P$.

(a) Give five explicit examples of idempotent 2×2-matrices.

(b) Find all idempotent 2×2-matrices.

(c) Let P be idempotent. Prove that $I - P$ is also idempotent.

(d) Let P be idempotent. Prove the formula $\mathcal{R}(I - P) = \mathcal{N}(P)$.

(e) Let P be idempotent. Prove the formula $\mathcal{N}(I - P) = \mathcal{R}(P)$.

(f) Let P be idempotent. Find $\mathcal{N}(P) + \mathcal{R}(P)$.

34. Show in general that the sum of two subspaces of a vector space is again a subspace.

35. Let $A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$.

(a) Find the echelon form of A, the basic variables, the free variables, and the solution to $Ax = 0$.

(b) For which b is the system $Ax = b$ solvable?

(c) Find the echelon form of A^T, the basic variables, the free variables, and the solution to $A^T x = 0$.

(d) For which b is the system $A^T x = b$ solvable?

36. Find all polynomials of degree two or less that pass through the points $(1, 1)$ and $(2, 2)$.